

Please read the following directions. (They are included in the directions I posted to the Canvas Announcements, so you can skip the following if you have already read those directions.)

- The exam is closed book and closed notes. No books, notes, or internet resources are allowed.
- A copy of the Table of Common Distributions from the back of our textbook is attached to the end of your exam.
- During the exam, you need only a supply of blank paper and writing implements (pens, pencils, erasers, etc). If you wish, you may use an ordinary scientific calculator (TI-86 or below is fine).
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations). **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, this work should be copied to your written solutions.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No “cooperation” is allowed.
- The exam has **6** problems and there are a total of **100** points.

Problem 1. A sample of four balls is chosen at random (without replacement) from an urn containing 9 red balls and 6 green balls. Let X be the number of red balls in the sample. Answer the following.

This problem combines features of exercises 2.3 and 1.51.

(a) (8%) Find $P(X = 2)$.

(b) (8%) Define $Y = 1/(X + 1)$ and let $F_Y(y)$ denote the cdf (cumulative distribution function) of Y . Find $F_Y(0.444)$.

Problem 2. (10%) Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. Assume the pdf f satisfies $f(x) > 0$ for $-\infty < x < \infty$. For fixed numbers a and b with $a < b$, define the function

$$g(x) = \begin{cases} \frac{f(x)}{F(b) - F(a)} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $g(x)$ is a pdf. (Give a detailed argument verifying all the requirements for g to be a pdf.)

This is similar to exercise 1.52.

Problem 3. A newly married couple agree that they would like to have both a boy and a girl child but cannot afford more than three children. So they plan to have two children and stop at that point if they have both a boy and a girl, but will go on and have a third child if they do not. Suppose the couple follow this plan. (Assume that the sex of different children is independent and that boys and girls have equal probability $1/2$. Also assume twins have probability zero.)

(a) (8%) Conditional on the couple having at least one boy, what is the probability all their children are boys?

This problem is similar to exercise 1.25 but somewhat more complicated.

Here are the various possibilities and their probabilities:

1/4 BG

1/4 GB

1/8 BBG

1/8 GGB

1/8 GGG

1/8 BBB

Answer = $\frac{1/8}{1-1/8} = 1/7$.

(b) (8%) The couple later wins the lottery and now can afford to have as many children as they want. So they change their plan and decide to keep having children until they have both a boy and a girl (and then they will stop). What is the probability they will have **more than** 5 children?

This is similar to exercise 1.26 but somewhat more complicated.

The answer is $(1/2)^4$. Let X denote the sex (B or G) of the couple's first child. They will continue having children until they have one whose sex is different from X . They will have more

than 5 children if their 2nd, 3rd, 4th, and 5th children all have sex X , which has probability $(1/2)^4$. Another way to state this: they will have more than 5 children if their first 5 children are $BBBBB$ or $GGGGG$, which has probability $2 \cdot (1/2)^5$.

Problem 4. Let X have pdf

$$f_X(x) = (x + 4)/18 \quad \text{for } -4 < x < 2$$

and define $Y = g(X)$ where the function $g(x)$ is given by

$$g(x) = \begin{cases} 2x^2 & \text{for } x \geq 0 \\ x^2 & \text{for } x < 0. \end{cases}$$

(a) (6%) Find $F_X(x)$, the cdf of X .

(b) (6%) Let A be the interval $A = (2, 8) = \{y : 2 < y < 8\}$. Find $g^{-1}(A)$.

(c) (6%) Use your answers to the previous parts to find $P_Y(A)$, where P_Y is the induced probability function of Y .

Parts (a), (b), (c) are similar to the example on pages 3-4 of notes3.pdf.

(d) (8%) Find $f_Y(y)$, the pdf of Y .

Part (d) is similar to exercise 2.7.

Problem 5. Let

$$Y = 2 \sum_{i=1}^{\infty} \frac{Z_i}{3^i}$$

where Z_1, Z_2, Z_3, \dots are independent with $P(Z_i = 0) = P(Z_i = 1) = 1/2$.

This is similar to exercise A3.

(a) (8%) Find EY .

(b) (8%) Find $\text{Var}(Y)$.

Problem 6. (16%) A fair dodecahedron (a 12-sided solid) has its sides labeled $1, 2, \dots, 12$. The dodecahedron is rolled five times. Let X_1, X_2, X_3, X_4, X_5 denote the observed sequence of values. Find the value of the conditional probability

$$P(X_1 < X_2 < X_3 < X_4 < X_5 \mid X_1 \leq X_2 \leq X_3 \leq X_4 \leq X_5)$$

This is similar to exercise B2.