## NWR = No Work Required

**For parts that are labeled NWR, you will get full credit just for stating the correct answer. You do NOT have to show any work or give any explanation.**

## Please read the following directions.

**(They are included in the directions I posted to the Canvas Announcements, so you can skip the following if you have already read those directions.)**

- The exam is closed book and closed notes. No books, notes, or internet resources are allowed.
- A copy of the Table of Common Distributions from the back of our textbook is attached to the end of your exam.
- During the exam, you need only a supply of blank paper and writing implements (pens, pencils, erasers, etc). If you wish, you may use an ordinary scientific calculator (TI-86 or below is fine).
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations). **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, this work should be copied to your written solutions.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **6** problems and there are a total of **100** points.

**Problem 1.** Suppose $X$ has density

$$f(x) = \begin{cases} \frac{2}{\sqrt{\pi}} e^{-x^2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \,. \end{cases}$$

*Similar to exercise 2.11.*

**(a)** (8%) Find $EX$.

**(b)** (8%) Find $\text{Var}(X)$.

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**Problem 2.** (10%) Suppose $X \sim$ Negative Binomial$(r, p)$ and $Y \sim$ Binomial$(n, p)$. (Here we use the lecture definition of the Negative Binomial distribution in which $Y$ is the number of trials needed to get $r$ successes. This is referred to as the "alternative form" in the appendix.)

Show that $F_X(n) = 1 - F_Y(r-1)$.

*This is equivalent to exercise 3.12, and also equivalent to the fact $P(T_r > n) = P(S_n < r)$ on page 17 of notes6.pdf. If we introduce alternate notation $Y = S_n$ and $X = T_r$, then $F_Y(r-1) = P(S_n < r)$ and $1 - F_X(n) = P(T_r > n)$. One correct solution is to quote the result $P(T_r > n) = P(S_n < r)$ from lecture and show this is equivalent to $F_X(n) = 1 - F_Y(r-1)$.*

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**Problem 3.** A *truncated* discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if $X$ has range 0, 1, 2, ... and the 0 class cannot be observed, the 0-*truncated* random variable $X_T$ has mass function (pmf)

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, 3, \ldots$$

Let $X_T$ be the 0-*truncated* random variable $X_T$ obtained starting from $X \sim$ Binomial$(n, p)$.

**(a)** (8%) Find the mean of $X_T$.

**(b)** (8%) Find the variance of $X_T$.

*This is similar to exercise 3.13.*

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**Problem 4.** Let $X$ be a random variable with moment generating function (mgf) given by

$$M(t) = 1 + \frac{pt}{1-t} \quad \text{for } t < 1\,.$$

where $p$ is some value in the range $0 \leq p \leq 1$. (Any value in this range leads to a valid mgf.)

**(a)** (8%) Find $EX$.

*Parts (a) and (b) require you to compute the first two moments from the mgf. There are a few exercises and lecture examples where moments are computed from the mgf. For example, exercise 2.33.*

**(b)** (8%) Find $\text{Var}(X)$.

**(c)** (6%, **NWR**) Let $X_1, X_2, \ldots, X_n$ be iid with the mgf $M(t)$ given above. Define

$$Y = \sum_{i=1}^{n} X_i .$$

Find $M_Y(t)$, the mgf of $Y$.

*The mgf of a sum of independent rv's is the product of their mgf's. So the answer is* $M_Y(t) = (M(t))^n$.

**(d)** (8%) Let $\beta > 0$. Find the limit of $M_Y(t)$ as $n \to \infty$, $p \to 0$, and $np \to \beta$. (For example, you could take $p_n = \beta/n$ for all $n$.)

*This is similar to exercise 3.15.*

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**Problem 5.** There are **four** snipers firing at the enemy from hidden positions. The enemy is constantly searching for the snipers' locations, and when a sniper's location is discovered, she is killed in a hail of bullets. Assume that the snipers begin their work at the same time and that the snipers' "lifetimes" (the time until they are discovered and killed) are i.i.d. exponential random variables with a mean of $\beta$ hours.

**(a)** (4%, **NWR**) Find the mean of the time until the last sniper dies.

**(b)** (4%, **NWR**) Find the variance of the time until the last sniper dies.

*This uses the general results on page 12 of notes7.pdf. Plugging* $k = 4$ *into these results is all that is needed. No other work is required.*

**(c)** (8%) The snipers are killed off one by one. Assume that, while $k$ snipers remain alive, they kill the enemy soldiers at an average rate of $ck^2$ deaths per hour. ($c$ is an arbitrary positive value.) What is the expected value of the total number of enemy soldiers killed by the four snipers?

*This is similar to exercise C4.*

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**Problem 6.** Let $X$ have density

$$f_X(x) = \begin{cases} \dfrac{\cos(x)}{B\sqrt{1-x^2}} & \text{for } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

where $B \approx 2.403939431$ .

*For a bounded rv, all moments exist and the mgf is finite for all values of* $t$. *This is stated on page 3 of notes5.pdf, and for moments on page 14 of notes4.pdf.*

**(a)** (6%) Does $EX^{13}$ exist? (In other words, is $EX^{13}$ a well-defined finite value?) Answer 'YES' or 'NO' and state a reason to justify your answer.

*A completely correct answer is "YES because* $X$ *is bounded."*

**(b)** (6%) Let $M_X(t)$ be the mgf of $X$. For what values of $t$ is $M_X(t)$ finite? State a reason to justify your answer.

*A completely correct answer is "The mgf is finite for* **all** $t$ *because* $X$ *is bounded.*