NWR = No Work Required **For parts that are labeled NWR, you will get full credit just for stating the correct answer. You do NOT have to show any work or give any explanation.**

Please read the following directions. **(They are included in the directions I posted to the Canvas Announcements, so you can skip the following if you have already read those directions.)**

- The exam is closed book and closed notes. No books, notes, or internet resources are allowed.
- A copy of the Table of Common Distributions from the back of our textbook is attached to the end of your exam.
- During the exam, you need only a supply of blank paper and writing implements (pens, pencils, erasers, etc). If you wish, you may use an ordinary scientific calculator (TI-86 or below is fine).
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations). **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, this work should be copied to your written solutions.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and there are a total of **100** points.

Problem 1. Suppose that the random variables X and Y satisfy

$$Y \sim \text{Discrete Uniform on } \{0, 1, \ldots, n\} \quad \textbf{(see note and table below)}$$
$$X \,|\, Y \sim \text{Beta}(Y+1\,,\, n-Y+1)$$

where n is a positive integer. Answer the following. Simplify your answers whenever possible.

(a) (8%) Find EX.

(b) (8%) Find $\text{Var}(X)$.

(c) (8%) Find the joint distribution of X and Y

(d) (8%) Find the marginal distribution of X. Specify the name of this distribution and the values of any parameters.

Hint for parts (c) and (d): It is useful to show that

$$\frac{1}{(n+1)B(y+1, n-y+1)} = \binom{n}{y}.$$

Note: The Discrete Uniform distribution in this problem differs slightly from the one in the appendix. Here is a table of useful facts for this distribution.

pmf	$f_Y(y) = \dfrac{1}{n+1}$ for $y = 0, 1, \ldots, n$
mean	$EY = \dfrac{n}{2}$
variance	$\text{Var}(Y) = \dfrac{n(n+2)}{12}$
an important expected value	$E[(Y+1)(n-Y+1)] = \dfrac{(n+2)(n+3)}{6}$

Problem 2. (10%) A casino has a game consisting of repeated rolls of a fair die. Players bet on the outcome. Two players (A and B) start betting at the same time. Player A always bets on the number 1 (that is, she wins whenever the die rolls a 1), and player B always bets on both 1 and 2 (that is, she wins whenever the die rolls either 1 or 2). Let X be the number of rolls until the first time player A wins, and Y be the number of rolls until the first time player B wins. Are X and Y independent? (Answer "Yes" or "No" and give a detailed justification of your answer.)

Problem 3. Suppose X_1 and X_2 are independent $N(0,1)$ random variables.

(a) (24%) Find the joint density of Y_1 and Y_2 where

$$Y_1 = X_1^2 + X_2^2 \quad \text{and} \quad Y_2 = \frac{X_1^2}{X_1^2 + X_2^2}.$$

(b) (6%) Show that Y_1 and Y_2 are independent.

Problem 4. (10%) Suppose X is a random variable with $EX = \mu_X$ and $\mathrm{Var}(X) = \sigma_X^2 < \infty$, and Y is a random variable with $P(Y = b) = 1$ where b is a constant. Find $\mathrm{Cov}(X, Y)$. Prove your answer.

All the remaining problems are NWR = No Work Required. You will get full credit just for stating the correct answer. You do NOT have to show any work or give any explanation.

Problem 5. (5%, **NWR**) Consider random variables (X, Y) with this joint pmf.

		X = 0	1	2
Y	0	1/9	1/9	1/9
	1	1/9	1/9	1/9
	2	1/9	1/9	1/9

Give a similar table for the joint pmf of random variables (X', Y') which satisfies:

$$X \stackrel{d}{=} X', \quad Y \stackrel{d}{=} Y', \quad X + Y \stackrel{d}{=} X' + Y', \quad \textbf{and also} \quad P(X' = 2, Y' = 0) = 0\,.$$

Problem 6. (8%, **NWR**) Let X and Y be i.i.d. with common density

$$f(x) = \begin{cases} c\,(1+x)^2(1-x) & \text{for } \mathbf{-1} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c = 3/4$. The density of $Z = X + Y$ can be found by integration:

$$f_Z(z) = \begin{cases} \displaystyle\int_A^B c^2(1+x)^2(1-x)(1+z-x)^2(1-z+x)\,dx & \text{for } 0 \le z < 2 \\ \displaystyle\int_D^E c^2(1+x)^2(1-x)(1+z-x)^2(1-z+x)\,dx & \text{for } \mathbf{-2} < z < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Where are the correct limits of integration above? Clearly specify the values of A, B, D, E. (These values may depend on z.)

Problem 7. (5%, **NWR**) Suppose $X_0, X_1, X_2, \ldots$ is a Markov chain with with initial distribution $\boldsymbol{a}$ and transition probability matrix $\boldsymbol{P}$. Give an expression which may be used to calculate the probability that the Markov chain is in **state 3** at **time 7**.

Table of Common Distributions

Do Not Mark !!

Discrete Distributions

Bernoulli(p)

pmf	$P(X=x\|p)=p^x(1-p)^{1-x};\quad x=0,1;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=p,\quad \mathrm{Var}\,X=p(1-p)$
mgf	$M_X(t)=(1-p)+pe^t$

Binomial(n,p)

pmf	$P(X=x\|n,p)=\binom{n}{x}p^x(1-p)^{n-x};\quad x=0,1,2,\ldots,n;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=np,\quad \mathrm{Var}\,X=np(1-p)$
mgf	$M_X(t)=[pe^t+(1-p)]^n$

notes Related to Binomial Theorem (Theorem 3.1.1). The *multinomial* distribution (Definition 4.6.1) is a multivariate version of the binomial distribution.

Discrete Uniform

pmf	$P(X=x\|N)=\frac{1}{N};\quad x=1,2,\ldots,N;\quad N=1,2,\ldots$
mean and variance	$\mathrm{E}X=\frac{N+1}{2},\quad \mathrm{Var}\,X=\frac{(N+1)(N-1)}{12}$
mgf	$M_X(t)=\frac{1}{N}\sum_{i=1}^{N}e^{it}$

Geometric(p)

pmf	$P(X=x\|p)=p(1-p)^{x-1};\quad x=1,2,\ldots;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=\frac{1}{p},\quad \mathrm{Var}\,X=\frac{1-p}{p^2}$
mgf	$M_X(t)=\frac{pe^t}{1-(1-p)e^t},\quad t<-\log(1-p)$

notes $Y=X-1$ is negative binomial$(1,p)$. The distribution is *memoryless*: $P(X>s|X>t)=P(X>s-t)$.

Hypergeometric

pmf	$P(X=x\|N,M,K)=\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}};\quad x=0,1,2,\ldots,K;$ $M-(N-K)\le x\le M;\quad N,M,K\ge 0$
mean and variance	$\mathrm{E}X=\frac{KM}{N},\quad \mathrm{Var}\,X=\frac{KM}{N}\frac{(N-M)(N-K)}{N(N-1)}$
notes	If $K<<M$ and N, the range $x=0,1,2,\ldots,K$ will be appropriate.

Negative binomial(r,p)

pmf	$P(X=x\|r,p)=\binom{r+x-1}{x}p^r(1-p)^x;\quad x=0,1,\cdots;\quad 0\le p\le 1$
mean and variance	$\mathrm{E}X=\frac{r(1-p)}{p},\quad \mathrm{Var}\,X=\frac{r(1-p)}{p^2}$
mgf	$M_X(t)=\left(\frac{p}{1-(1-p)e^t}\right)^r,\quad t<-\log(1-p)$

notes An alternate form of the pmf is given by $P(Y=y|r,p)=\binom{y-1}{r-1}p^r(1-p)^{y-r}$, $y=r,r+1,\ldots$. The random variable $Y=X+r$. The negative binomial can be derived as a gamma mixture of Poissons. (See Exercise 4.34.)

Poisson(λ)

pmf	$P(X=x\|\lambda)=\frac{e^{-\lambda}\lambda^x}{x!};\quad x=0,1,\ldots;\quad 0\le\lambda<\infty$
mean and variance	$\mathrm{E}X=\lambda,\quad \mathrm{Var}\,X=\lambda$
mgf	$M_X(t)=e^{\lambda(e^t-1)}$

Continuous Distributions

Beta(α, β)

pdf $f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1, \quad \alpha > 0, \quad \beta > 0$

mean and variance $\mathrm{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\, X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

mgf $M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$

notes The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

Cauchy(θ, σ)

pdf $f(x|\theta,\sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty, \quad \sigma > 0$

mean and variance do not exist

mgf does not exist

notes Special case of Student's t, when degrees of freedom $= 1$. Also, if X and Y are independent n(0, 1), X/Y is Cauchy.

Chi squared

pdf $f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}; \quad 0 \le x < \infty; \quad p = 1, 2, \ldots$

mean and variance $\mathrm{E}X = p, \quad \mathrm{Var}\, X = 2p$

mgf $M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2}, \quad t < \frac{1}{2}$

notes Special case of the gamma distribution.

Double exponential(μ, σ)

pdf $f(x|\mu,\sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = 2\sigma^2$

mgf $M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad |t| < \frac{1}{\sigma}$

notes Also known as the *Laplace* distribution.

Exponential(β)

pdf $f(x|\beta) = \frac{1}{\beta}e^{-x/\beta}, \quad 0 \le x < \infty, \quad \beta > 0$

mean and variance $\mathrm{E}X = \beta, \quad \mathrm{Var}\, X = \beta^2$

mgf $M_X(t) = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}$

notes Special case of the gamma distribution. Has the *memoryless* property. Has many special cases: $Y = X^{1/\gamma}$ is *Weibull*, $Y = \sqrt{2X/\beta}$ is *Rayleigh*, $Y = \alpha - \gamma \log(X/\beta)$ is *Gumbel*.

F

pdf $f(x|\nu_1, \nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{(\nu_1-2)/2}}{\left(1+\left(\frac{\nu_1}{\nu_2}\right)x\right)^{(\nu_1+\nu_2)/2}}; \quad 0 \le x < \infty;$
$\nu_1, \nu_2 = 1, \ldots$

mean and variance $\mathrm{E}X = \frac{\nu_2}{\nu_2-2}, \quad \nu_2 > 2,$

$\mathrm{Var}\, X = 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}, \quad \nu_2 > 4$

moments (mgf does not exist) $\mathrm{E}X^n = \frac{\Gamma\left(\frac{\nu_1+2n}{2}\right)\Gamma\left(\frac{\nu_2-2n}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_2}{\nu_1}\right)^n, \quad n < \frac{\nu_2}{2}$

notes Related to chi squared ($F_{\nu_1,\nu_2} = \left(\frac{\chi^2_{\nu_1}}{\nu_1}\right) / \left(\frac{\chi^2_{\nu_2}}{\nu_2}\right)$, where the χ^2s are independent) and t ($F_{1,\nu} = t^2_\nu$).

Gamma(α, β)

pdf $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 \le x < \infty, \quad \alpha,\ \beta > 0$

mean and variance $\mathrm{E}X = \alpha\beta, \quad \mathrm{Var}\, X = \alpha\beta^2$

mgf $M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$

notes Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2, \beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).

Logistic(μ, β)

pdf $f(x|\mu,\beta) = \frac{1}{\beta}\frac{e^{-(x-\mu)/\beta}}{[1+e^{-(x-\mu)/\beta}]^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty,$ $\beta > 0$

mean and variance $\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \frac{\pi^2\beta^2}{3}$

mgf $M_X(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t), \quad |t| < \frac{1}{\beta}$

notes The cdf is given by $F(x|\mu,\beta) = \frac{1}{1+e^{-(x-\mu)/\beta}}$.

Lognormal(μ, σ^2)

pdf $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/(2\sigma^2)}}{x}, \quad 0 \le x < \infty, \quad -\infty < \mu < \infty,$ $\sigma > 0$

mean and variance $\mathrm{E}X = e^{\mu+(\sigma^2/2)}, \quad \mathrm{Var}\, X = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$

moments (mgf does not exist) $\mathrm{E}X^n = e^{n\mu+n^2\sigma^2/2}$

notes Example 2.3.5 gives another distribution with the same moments.

Normal(μ, σ^2)

pdf $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty,$ $-\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $\mathrm{E}X = \mu, \mathrm{Var}\, X = \sigma^2$

mgf $M_X(t) = e^{\mu t+\sigma^2t^2/2}$

notes Sometimes called the *Gaussian* distribution.

Pareto(α, β)

pdf $f(x|\alpha,\beta) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \quad \alpha < x < \infty, \quad \alpha > 0, \quad \beta > 0$

mean and variance $\mathrm{E}X = \frac{\beta\alpha}{\beta-1}, \quad \beta > 1,$

$\mathrm{Var}\, X = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}, \quad \beta > 2$

mgf does not exist

t

pdf	$f(x\|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+\left(\frac{x^2}{\nu}\right)\right)^{(\nu+1)/2}}, \quad -\infty < x < \infty, \quad \nu = 1, \ldots$
mean and variance	$\mathrm{E}X = 0, \quad \nu > 1,$ $\mathrm{Var}\,X = \frac{\nu}{\nu-2}, \quad \nu > 2$
moments (mgf does not exist)	$\mathrm{E}X^n = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{\nu-n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \nu^{n/2}$ if $n < \nu$ and even, $\mathrm{E}X^n = 0$ if $n < \nu$ and odd.
notes	Related to F $(F_{1,\nu} = t_\nu^2)$.

Uniform(a, b)

pdf	$f(x\|a, b) = \frac{1}{b-a}, \quad a \le x \le b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\,X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$
notes	If $a = 0$ and $b = 1$, this is a special case of the beta ($\alpha = \beta = 1$).

Weibull(γ, β)

pdf	$f(x\|\gamma, \beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}, \quad 0 \le x < \infty, \quad \gamma > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \beta^{1/\gamma}\Gamma\left(1 + \frac{1}{\gamma}\right), \quad \mathrm{Var}\,X = \beta^{2/\gamma}\left[\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right)\right]$
moments	$\mathrm{E}X^n = \beta^{n/\gamma}(1 + \frac{n}{\gamma})$

notes The mgf exists only for $\gamma \ge 1$. Its form is not very useful. A special case is exponential ($\gamma = 1$).