NWR = No Work Required **For parts that are labeled NWR, you will get full credit just for stating the correct answer. You do NOT have to show any work or give any explanation.**

Please read the following directions. **(They are included in the directions I posted to the Canvas Announcements, so you can skip the following if you have already read those directions.)**

- The exam is closed book and closed notes. No books, notes, or internet resources are allowed.
- A copy of the Table of Common Distributions from the back of our textbook is attached to the end of your exam.
- During the exam, you need only a supply of blank paper and writing implements (pens, pencils, erasers, etc). If you wish, you may use an ordinary scientific calculator (TI-86 or below is fine).
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations). **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, this work should be copied to your written solutions.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and there are a total of **100** points.

Problem 1. Suppose that the random variables X and Y satisfy

$$\begin{aligned} Y &\sim \text{Discrete Uniform on } \{0, 1, \ldots, n\} \quad (\textbf{see note and table below}) \\ X \,|\, Y &\sim \text{Beta}(Y + 1\,,\; n - Y + 1) \end{aligned}$$

where n is a positive integer. Answer the following. Simplify your answers whenever possible.

(a) (8%) Find EX.

Answer: $EX = 1/2$

Students should simplify everything in all the parts of this problem in order to get full credit. The numbers have been arranged so this is not difficult.

(b) (8%) Find $\text{Var}(X)$.

Answer: $\mathit{Var}(X) = \dfrac{1}{12}$

(c) (8%) Find the joint distribution of X and Y

Answer: $f(x, y) = \binom{n}{y} x^y (1 - x)^{n-y}$ *for* $0 < x < 1,\ y = 0, 1, \ldots, n.$

Students must specify the support to get full credit, but maybe it is only worth one point.

(d) (8%) Find the marginal distribution of X. Specify the name of this distribution and the values of any parameters.

Answer: $f_X(x) = 1$ *for* $0 < x < 1$ *so that* $X \sim \mathit{Uniform}(0, 1)$.

Hint for parts (c) and (d): It is useful to show that

$$\frac{1}{(n + 1)B(y + 1, n - y + 1)} = \binom{n}{y}.$$

Note: The Discrete Uniform distribution in this problem differs slightly from the one in the appendix. Here is a table of useful facts for this distribution.

$$\begin{array}{rl} \text{pmf} & f_Y(y) = \dfrac{1}{n+1} \quad \text{for} \quad y = 0, 1, \ldots, n \\ \text{mean} & EY = \dfrac{n}{2} \\ \text{variance} & \text{Var}(Y) = \dfrac{n(n+2)}{12} \\ \textbf{an important expected value} & E[(Y + 1)(n - Y + 1)] = \dfrac{(n+2)(n+3)}{6} \end{array}$$

Problem 2. (10%) A casino has a game consisting of repeated rolls of a fair die. Players bet on the outcome. Two players (A and B) start betting at the same time. Player A always bets on the number 1 (that is, she wins whenever the die rolls a 1), and player B always bets on both 1 and 2 (that is, she wins whenever the die rolls either 1 or 2). Let X be the number of rolls until the first time player A wins, and Y be the number of rolls until the first time player B wins. Are X and Y independent? (Answer "Yes" or "No" and give a detailed justification of your answer.)

Answer: No, X and Y are **not** *independent. They canNOT be independent because the support of the joint distribution is NOT a product set. But to get full credit you must show this. Here is an argument: Whenever A wins, B also wins. This implies $Y \le X$. Therefore, for example, $P(X = 2, Y = 4) = 0$. But clearly $X = 2$ and $Y = 4$ are possible values of X and Y, that is, $P(X = 2) > 0$ and $P(Y = 4) > 0$. Thus $P(X = 2, Y = 4) = 0 \neq P(X = 2)P(Y = 4)$ which directly shows X and Y are not independent (and also shows the support is not a product set). Obviously, we can construct the same sort of example using other pairs of values.*

Problem 3. Suppose X_1 and X_2 are independent $N(0, 1)$ random variables.

This problem is somewhat similar to both 4.20 and 4.19(b).

(a) (24%) Find the joint density of Y_1 and Y_2 where

$$Y_1 = X_1^2 + X_2^2 \quad \text{and} \quad Y_2 = \frac{X_1^2}{X_1^2 + X_2^2} .$$

This problem can be worked in two different ways.

First approach: (Similar to exercise 4.20) View this as a 4-to-1 transformation. Each of the four quadrants in the x_1-x_2 plane is mapped in a 1-to-1 fashion to the strip $(0, \infty) \times (0, 1)$. Each quadrant gives an equal contribution to the joint density, so the joint density of (Y_1, Y_2) is four times what you would get from the first quadrant alone.

Second approach: Define $Z_1 = X_1^2$ and $Z_2 = X_2^2$. Then Z_1 and Z_2 are iid χ^2_1 random variables (the same as Gamma($\alpha = 1/2, \beta = 2$)) and

$$Y_1 = Z_1 + Z_2 \quad \textit{and} \quad Y_2 = \frac{Z_1}{Z_1 + Z_2} .$$

This bivariate transformation is 1-to-1. The problem is now very similar to exercise 4.19(b), and essentially the same solution works.

(b) (6%) Show that Y_1 and Y_2 are independent.

It is easiest to use the lemma stated in lecture: the joint density $f(y_1, y_2)$ can be expressed in the form $cg(y_1)h(y_2)$ for $(y_1, y_2) \in A \times B$. Therefore Y_1 and Y_2 are independent.

Problem 4. (10%) Suppose X is a random variable with $EX = \mu_X$ and $\text{Var}(X) = \sigma_X^2 < \infty$, and Y is a random variable with $P(Y = b) = 1$ where b is a constant. Find $\text{Cov}(X, Y)$. Prove your answer.

This is Exercise 4.41 in a slightly disguised form.
The result Cov$(X, b) = 0$ where b is a constant was stated in lecture, and students should get partial credit for quoting this result. But they should give a proof to get full credit.

All the remaining problems are NWR = No Work Required. You will get full credit just for stating the correct answer. You do NOT have to show any work or give any explanation.

Problem 5. (5%, **NWR**) Consider random variables (X, Y) with this joint pmf.

		X = 0	1	2
Y	0	1/9	1/9	1/9
	1	1/9	1/9	1/9
	2	1/9	1/9	1/9

Give a similar table for the joint pmf of random variables (X', Y') which satisfies:

$$X \stackrel{d}{=} X', \quad Y \stackrel{d}{=} Y', \quad X+Y \stackrel{d}{=} X'+Y', \quad \textbf{and also} \quad P(X'=2, Y'=0) = 0\,.$$

The answer is:

		X' = *0*	*1*	*2*
Y'	*0*	*1/9*	*2/9*	*0*
	1	*0*	*1/9*	*2/9*
	2	*2/9*	*0*	*1/9*

This problem is based on the example indep_binomials_perturbed.pdf.

Problem 6. (8%, **NWR**) Let X and Y be i.i.d. with common density

$$f(x) = \begin{cases} c\,(1+x)^2(1-x) & \text{for } \mathbf{-1} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c = 3/4$. The density of $Z = X + Y$ can be found by integration:

$$f_Z(z) = \begin{cases} \displaystyle\int_A^B c^2(1+x)^2(1-x)(1+z-x)^2(1-z+x)\,dx & \text{for } 0 \le z < 2 \\ \displaystyle\int_D^E c^2(1+x)^2(1-x)(1+z-x)^2(1-z+x)\,dx & \text{for } \mathbf{-2} < z < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Where are the correct limits of integration above? Clearly specify the values of A, B, D, E. (These values may depend on z.)

Answer: The limits are $A = z-1,\ B = 1,\ D = -1,\ E = z+1$.
This problem is similar to the lecture example on page 21 of notes10.pdf.

Problem 7. (5%, **NWR**) Suppose $X_0, X_1, X_2, \ldots$ is a Markov chain with with initial distribution $\boldsymbol{a}$ and transition probability matrix $\boldsymbol{P}$. Give an expression which may be used to calculate the probability that the Markov chain is in **state 3** at **time 7**.

The answer is $(aP^7)_3$.
This uses the result on page 13 of notes13_markov_chains.pdf. Here $(aP^7)_3$ *is the third coordinate in the row vector* aP^7 *where* P^7 *is the matrix power.*