TEST #1
STA 5326 **Name:**____________________________________
September 27, 2021

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems. **No credit is given without work or explanation!** This even includes the counting problems. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **10** pages. There are a total of **100** points.

Problem 1. Suppose X has density (pdf) $f(x) = cx^2$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

(a) (5%) What is the value of the constant c?

(b) (5%) Calculate $P(1/3 < X < 2/3)$.

[**Problem 1 continued**]

(c) (5%) Calculate $E(\log X)$.

(d) (5%) Calculate $\mathrm{Var}(X^3)$.

Problem 2. (18%) Let X have density $f_X(x) = ce^x$ for $-1 < x < 1$ where $c = e/(e^2 - 1)$, and define

$$Y = \begin{cases} X^2 & \text{if } X \le 0 \\ X^{1/3} & \text{if } X > 0\,. \end{cases}$$

Find the density (pdf) of Y. (You may leave c in your answer.)

Problem 3. (18%) Prove the principle of inclusion-exclusion for the case of $k = 4$ events. You may use the result for $k = 2$ and $k = 3$ events in your proof.

Problem 4. 100 students are guests at a party; 25 of them are freshmen, 25 are sophomores, 25 are juniors, and 25 are seniors. 20 different gifts will be given as prizes to the party guests; 4 of these gifts are "really great", 7 are "pretty nice", and 9 are just "OK".

The gifts are given as follows. The 100 names of the guests are written on 100 tickets and placed in a bowl (bowl A). Descriptions of the 20 gifts are written on 20 tickets and placed in another bowl (bowl B). A name is drawn at random from bowl A and a gift description is drawn at random from bowl B; the person whose name is drawn receives the described gift. Then another name is drawn from bowl A and another gift description from bowl B; the person whose name is drawn receives the described gift, etc. This continues until bowl B is empty. (The random sampling from both bowls is done withOUT replacement.)

Answer the following.

To receive full credit, you must explain/justify your answers!

(a) (8%) What is the probability that exactly 5 freshmen, 5 sophomores, 5 juniors, and 5 seniors receive gifts?

[**Problem 4 continued**]

(b) (8%) What is the probability that none of the first 10 gifts given are “really great”?

(c) (4%) What is the probability that all of the “really great” gifts are won by freshmen?

Problem 5. (6%) Consider the Gambler's Ruin with a fair coin as described in lecture. (The gambler wins a dollar every time the coin comes up heads, and loses a dollar when it is tails.) We showed in lecture that the probability of reaching the goal of g dollars starting from an initial fortune of z dollars (with $0 < z < g$) is $\psi(z) = z/g$.

Find the probability the first three tosses were heads, given that the gambler reached the goal. (Assume $z + 3 \leq g$.)

Problem 6. (4%) Suppose X has the density

$$f_X(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad \text{for } 0 < x < 1$$

Find a good approximation to $P(0.5 - 10^{-6} < X < 0.5 + 10^{-6})$.

Problem 7. (6%) A dart is tossed uniformly at random at a circular target with radius 3 which has its center at the origin $(0, 0)$. Let X be the distance of the dart from the origin. Find the cumulative distribution function (cdf) of X.

Problem 8. (8%) Let X have density $f_X(x) = x/8$ for $0 < x < 4$. Define

$$Y = \begin{cases} 3X & \text{for} \quad 0 < X \le 1 \\ 2(X-1) & \text{for} \quad 1 < X \le 2 \\ 0.5(X-2)^3 & \text{for} \quad 2 < X \le 3 \\ 0.5\sqrt{X-3} & \text{for} \quad 3 < X < 4\,. \end{cases}$$

Find $f_Y(1)$, the density of Y evaluated at 1.