TEST #1
STA 5326 **Name:**____________________________________
September 27, 2021

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems. **No credit is given without work or explanation!** This even includes the counting problems. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **10** pages. There are a total of **100** points.

General Remarks on grading:

I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).

I usually do not deduct points for copying errors (unless they make the answer ridiculous).

I do deduct points for algebra and calculus errors, but as long as the student's solution is basically correct (except for the algebra or calculus mistakes), I try to give them at least half the credit.

However, if an algebra or calculus mistake is bad enough to transform the problem so that the solution after the point of the mistake no longer resembles the correct solution, then the student only gets credit for the work up to the point of the mistake.

In general, if a student makes a mistake which completely changes the problem, and then the student correctly solves the changed problem, they only get credit up to the point of the mistake; they don't get credit for correctly solving the wrong problem.

Problem 1. Suppose X has density (pdf) $f(x) = cx^2$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

This is similar to exercise A2.

(a) (5%) What is the value of the constant c?

(b) (5%) Calculate $P(1/3 < X < 2/3)$.

[**Problem 1 continued**]

(c) (5%) Calculate $E(\log X)$.

(d) (5%) Calculate $\mathrm{Var}(X^3)$.

Problem 2. (18%) Let X have density $f_X(x) = ce^x$ for $-1 < x < 1$ where $c = e/(e^2 - 1)$, and define

$$Y = \begin{cases} X^2 & \text{if } X \le 0 \\ X^{1/3} & \text{if } X > 0\,. \end{cases}$$

Find the density (pdf) of Y. (You may leave c in your answer.)

This is similar to exercise 2.6.

Problem 3. (18%) Prove the principle of inclusion-exclusion for the case of $k = 4$ events. You may use the result for $k = 2$ and $k = 3$ events in your proof.

This is exercise B5.1. Most students will probably give a solution similar to the posted solution. But some may choose to use indicator rv's and give a solution similar to page 24 of notes4.pdf; this is perfectly fine.

Problem 4. 100 students are guests at a party; 25 of them are freshmen, 25 are sophomores, 25 are juniors, and 25 are seniors. 20 different gifts will be given as prizes to the party guests; 4 of these gifts are "really great", 7 are "pretty nice", and 9 are just "OK".

The gifts are given as follows. The 100 names of the guests are written on 100 tickets and placed in a bowl (bowl A). Descriptions of the 20 gifts are written on 20 tickets and placed in another bowl (bowl B). A name is drawn at random from bowl A and a gift description is drawn at random from bowl B; the person whose name is drawn receives the described gift. Then another name is drawn from bowl A and another gift description from bowl B; the person whose name is drawn receives the described gift, etc. This continues until bowl B is empty. (The random sampling from both bowls is done withOUT replacement.)

Answer the following.

To receive full credit, you must explain/justify your answers!

Some notation used in the solutions:

Let $(n)_r = n(n-1)(n-2)\cdots(n-r+1)$, the number of possible ordered arrangements (permutations) of r objects chosen from n.

If $n, j_1, j_2, \ldots, j_k$ are nonnegative integers satisfying $n = j_1 + j_2 + \cdots + j_k$, then we define the multinomial coefficient

$$\binom{n}{j_1, j_2, \ldots, j_k} = \frac{n!}{j_1! j_2! \ldots j_k!} = \binom{n}{j_1}\binom{n-j_1}{j_2}\binom{n-j_1-j_2}{j_3}\cdots\binom{n-j_1-j_2-\cdots-j_{k-2}}{j_{k-1}}$$

The multinomial coefficient is the number of ways to divide n objects into k groups of sizes j_1, j_2, $\ldots$, j_k. The multinomial coefficient was implicitly used in a couple of homework solutions where it was written as a product of binomial coefficients as in the rightmost expression above.

(a) (8%) What is the probability that exactly 5 freshmen, 5 sophomores, 5 juniors, and 5 seniors receive gifts?

This is similar to exercise 1.22(a).

Let C be the event that exactly 5 freshmen, 5 sophomores, 5 juniors, and 5 seniors receive gifts. Here we are interested only in who receives gifts, and not in what gifts they receive. So we focus on the draws from Bowl A and ignore Bowl B.

The answer is $\dfrac{\binom{25}{5}^4}{\binom{100}{20}} = \dfrac{[(25)_5]^4}{(100)_{20}} \cdot \dbinom{20}{5,5,5,5}$

The expression on the left is obtained by viewing the 20 persons selected from Bowl A as an **un***ordered set; there are $\binom{100}{20}$ equally likely possibilities; this is $\#(\Omega)$ and gives the denominator above. We can construct the unordered sets containing exactly 5 each of freshman, sophomores, juniors, and seniors in four steps: select 5 freshmen, select 5 sophomores, select 5 juniors, and select 5 seniors. Each of the steps can be done in $\binom{25}{4}$ ways. The product $\binom{25}{4}^4$ is $\#(C)$ and is the numerator above.*

The expression on the right is obtained by viewing the 20 persons as being ordered. There are now $\#(\Omega) = (100)_{20}$ equally likely possibilities, which is the denominator on the right above. Let

F = Freshman, S = Sophomore, J = Junior, and E = sEnior. We can construct an arrangement like $FFFFFSSSSSJJJJJEEEEE$ in 4 steps: make an ordered selection of 5 freshmen, follow this with an ordered selection of 5 sophomores, follow this with an ordered selection of 5 juniors, and follow this with an ordered selection of 5 seniors. Each of the steps can be done in $(25)_5$ ways. Taking the product and dividing by $\#(\Omega)$ gives the probability that the draws from Bowl A have the order $FFFFFSSSSSJJJJJEEEEE$. But this is only one possible ordering. Any rearrangement of the 5 F's, 5 S's, 5 J's, and 5 E's has an equal probability, so we multiply the answer by the number of such rearrangements, which is the multinomial coefficient $\binom{20}{5,5,5,5}$ appearing on the right above. This gives our final answer.

Both approaches (unordered and ordered) give the same answer. The unordered approach is simpler in this case.

Note that we can also compute the probability of the draws having the ordering $FFFFFSSSSSJJJJJEEEEE$ using an urn model, leading to an answer having the same form as the righthand (ordered) approach above.

[**Problem 4 continued**]

(b) (8%) What is the probability that none of the first 10 gifts given are "really great"?

This is similar to 1.22(b)

In this part we are not concerned with who gets the gifts, only with the order in which they are given. So we can focus on the draws from Bowl B and completely ignore Bowl A.

The answer can be written in several ways, each with a different argument leading to it:

$$\frac{\binom{16}{10}}{\binom{20}{10}} = \frac{16}{20} \cdot \frac{15}{19} \cdot \frac{14}{18} \cdot \frac{13}{17} \cdot \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} \cdot \frac{9}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{20 \cdot 19 \cdot 18 \cdot 17}$$

The leftmost answer has the simplest argument. The first 10 gifts drawn from Bowl B may be viewed as an **un***ordered set of 10 gifts selected from the 20 gifts. There are* $\#(\Omega) = \binom{20}{10}$ *equally likely possibilites, which is the denominator of the answer. We may construct a set of 10 gifts without any "really great" gifts by choosing all 10 from the 16 gifts which are "pretty nice" or just "OK". This can be done in* $\binom{16}{10}$ *ways, which is the numerator of the answer.*

(If anyone gives you one of the other forms of the answer, I can send you the arguments for those.)

(c) (4%) What is the probability that all of the "really great" gifts are won by freshmen?

The answer may be written in several ways:

$$\frac{\binom{25}{4}}{\binom{100}{4}} = \frac{25 \cdot 24 \cdot 23 \cdot 22}{100 \cdot 99 \cdot 98 \cdot 97} = \frac{20! \cdot (25)_4 \cdot (96)_{16}}{20!(100)_{20}}$$

The leftmost answer has the simplest story. Consider the random unordered set of 4 people who receive the four "really great" gifts. All the guests are treated the same in the gift-giving process, so it is clear that all sets of 4 people are equally likely to be the lucky ones. That is, there are $\binom{100}{4}$ *equally likely possibilities. In* $\binom{25}{4}$ *of these possibilities, all 4 of the lucky ones are freshmen. So the ratio (the leftmost expression) is the desired probability.*

Problem 5. (6%) Consider the Gambler's Ruin with a fair coin as described in lecture. (The gambler wins a dollar every time the coin comes up heads, and loses a dollar when it is tails.) We showed in lecture that the probability of reaching the goal of g dollars starting from an initial fortune of z dollars (with $0 < z < g$) is $\psi(z) = z/g$.

Find the probability the first three tosses were heads, given that the gambler reached the goal. (Assume $z + 3 \leq g$.)

See page 35 of notes1.pdf

Problem 6. (4%) Suppose X has the density

$$f_X(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad \text{for } 0 < x < 1$$

Find a good approximation to $P(0.5 - 10^{-6} < X < 0.5 + 10^{-6})$.

Use the heuristic interpretation of densities on page 13 of notes3.pdf.

Problem 7. (6%) A dart is tossed uniformly at random at a circular target with radius 3 which has its center at the origin $(0, 0)$. Let X be the distance of the dart from the origin. Find the cumulative distribution function (cdf) of X.

See page 20 of notes2.pdf and also page 5 of notes1.pdf.

Problem 8. (8%) Let X have density $f_X(x) = x/8$ for $0 < x < 4$. Define

$$Y = \begin{cases} 3X & \text{for} \quad 0 < X \le 1 \\ 2(X-1) & \text{for} \quad 1 < X \le 2 \\ 0.5(X-2)^3 & \text{for} \quad 2 < X \le 3 \\ 0.5\sqrt{X-3} & \text{for} \quad 3 < X < 4\,. \end{cases}$$

Find $f_Y(1)$, the density of Y evaluated at 1.

The easiest approach is to use the formula on page 24 of notes3.pdf. If we let g be the function defined in the braces above so that $Y = g(X)$, then $g^{-1}(\{1\}) = \{1/3, 3/2\}$ so that $f_Y(1) = f_X(1/3)(1/3) + f_X(3/2)(1/2) = \dfrac{1}{3 \cdot 8 \cdot 3} + \dfrac{3}{2 \cdot 8 \cdot 2}$.