TEST #3
STA 5326 **Name:**____________________________________
December 9, 2021

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems (except those designated as "no work required"). **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **6** problems and **10** pages. There are a total of **100** points.

Problem 1. Suppose that

$$\begin{aligned} X \sim & \text{Poisson}(\lambda) \\ Y \mid X \sim & \text{Binomial}(X, p) \,. \end{aligned}$$

This problem resembles the homework exercises with two-stage (hierarchical) models such as 4.31 and 4.32(a).

(a) (7%) Find EY

Use $EY = EE(Y \mid X)$ *to obtain* $EY = \lambda p \,.$

(b) (10%) Find $\text{Var}(Y)$.

Use $\mathit{Var}(Y) = E\,\mathit{Var}(Y \mid X) + \mathit{Var}(E(Y \mid X))$ *to obtain* $\mathit{Var}(Y) = \lambda p \,.$

[**Problem 1 continued**]

(c) (7%) Find the joint distribution of X and Y. (Specify the support.)

$f_{X,Y}(x,y) = f_X(x) f_{Y\,|\,X}(y\,|\,x) = \dfrac{\lambda^x e^{-\lambda}}{x!}\dbinom{x}{y} p^y (1-p)^{x-y}$ *for* $x = 0, 1, 2, \ldots$ *and* $y = 0, 1, \ldots, x$.
This can be algebraically re-expressed in various ways.

(d) (6%) Find the marginal distribution of Y.

Hint: Use $e^z = \displaystyle\sum_{i=0}^{\infty} \frac{z^i}{i!}$.

$$\begin{aligned} f_Y(y) &= \sum_x f_{X,Y}(x,y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}\binom{x}{y} p^y (1-p)^{x-y} \\ &= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{(\lambda(1-p))^{x-y}}{(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} \cdot e^{\lambda(1-p)} \\ &= \frac{(\lambda p)^y}{y!} e^{-\lambda p}, \quad y = 0, 1, 2, \ldots \end{aligned}$$

after using the Hint. We recognize this as the pmf of a Poisson(λp) distribution. This gives a check on the answers to parts (a) and (b).

Problem 2. Let X and Y be iid with density

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \quad \text{for } x > 0.$$

Define $U = X + Y$ and $W = X - Y$.

(a) (14%) Find the joint density of U and W. (Make sure to carefully specify the support.)

The inverse transformation is $X = (U+W)/2$, $Y = (U-W)/2$ which has Jacobian $J = -1/2$. Since the rv's X and Y are positive, we know that $(U+W)/2 > 0$ and $(U-W)/2 > 0$ which implies $U > 0$, $W > -U$ and $W < U$ so that the support of (U, W) is $\{(u, w) : u > 0,\ -u < w < u\}$. This gives the joint density

$$\begin{aligned} f_{U,W}(u, w) &= f_X((u+w)/2) f_Y((u-w)/2) \cdot \frac{1}{2} = \frac{1}{\beta} e^{-\frac{(u+w)}{2\beta}} \cdot \frac{1}{\beta} e^{-\frac{(u-w)}{2\beta}} \cdot \frac{1}{2} \quad \textit{for } u > 0,\ -u < w < u \\ &= \frac{e^{-u/\beta}}{2\beta^2} \quad \textit{for } u > 0,\ -u < w < u\,. \end{aligned}$$

[**Problem 2 continued**]

(b) (6%) Are U and W independent? (Answer "Yes" or "No" and then **prove your answer**.)

No, *U and W are NOT independent. They canNOT be independent since the support $\{(u,w) : f_{U,W}(u,w) > 0\} = \{(u,w) : u > 0$ and $-u < w < u\}$ is NOT a product set. This is a sufficient proof.*

Other proofs are possible such as showing that the joint pdf of (U,W) is NOT the product of the marginal pdf's. But for this you need to find the marginal pdf's. They are given by:

$$f_U(u) = \frac{ue^{-u/\beta}}{\beta^2} \quad \textit{for } u > 0,$$
$$f_W(w) = \frac{e^{-|w|/\beta}}{2\beta} \quad \textit{for } -\infty < w < \infty\,.$$

These marginals can be derived from the joint density $f_{U,W}$, but actually we know them already. Since $U = X + Y$ and X, Y are independent exponential rv's, by the closure property for sums of gamma rv's we know that $U \sim$ Gamma$(\alpha = 2, \beta)$. Also, $W = X - Y$ and very early in the the course we found that the difference of two iid exponential rv's has a double exponential distribution. *See pages 15-16 of notes5.pdf.*

Problem 3. Let X and Y have joint pdf defined by

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{6x^2y}{(1-x)^2} & \text{if } x > 0,\ y > 0, \text{ and } x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

This problem is (sort of) a combination of problem 4.4 and the lecture example on pages 6-7 and 12-13 of notes9.pdf.

(a) (7%) Find the marginal density of X.

$f_X(x) = 3x^2$ *for* $0 < x < 1$.

(b) (7%) Find $F_{X,Y}(1/2, 3/2)$, the joint **c**df of X and Y evaluated at the point $(1/2, 3/2)$.

After drawing a picture showing where the point $(1/2, 3/2)$ *lies relative to the support* $\{(x,y) : x > 0, y > 0, x + y < 1\}$, *it is clear that* $F_{X,Y}(1/2, 3/2) = F_X(1/2) = x^3|_{x=1/2} = 1/8$.

[**Problem 3 continued**]

(c) (7%) Find $f_{Y|X}(y|x)$, the conditional density of Y given $X = x$. (For what values of x is this defined?)

The conditional density is only defined for $0 < x < 1$*. For* $0 < x < 1$*, it is given by* $f_{Y|X}(y|x) = \dfrac{2y}{(1-x)^2}$ *for* $0 < y < 1 - x$ *(and zero otherwise).*

(d) (5%) Find $E(Y|X = x)$. (For what values of x is this defined?)

The conditional expectation is only defined for $0 < x < 1$*. In this range it is given by*

$$E(Y|X = x) = \frac{2}{3}(1 - x)\,.$$

Problem 4. (12%) Suppose X, Y, Z are iid with mean μ and variance $\sigma^2 > 0$. Find the correlation between $XY + 5$ and $YZ - 2$.

The correlation is $\dfrac{\mu^2}{2\mu^2 + \sigma^2}$. *The constants* $+5$ *and* -2 *can be dropped immediately according to the discussion in the lecture notes. This problem is somewhat similar to exercise 4.42.*

The remaining problems require no work. You will receive full credit just for stating the correct answer. (Circle your answer if there is any doubt.) No partial credit is given.

Problem 5. (4%) Let X and Y be independent random variables with densities

$$f_X(x) = \frac{1}{\pi(1+x^2)} \quad \text{for} \quad -\infty < x < \infty \quad \text{and}$$
$$f_Y(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \quad \text{for} \quad -\infty < y < \infty\,.$$

Define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0\,. \end{cases}$$

What is the density of Z? (Give a precise formula.)

This problem is similar to 4.47(a).

Z has the same density as X which is $f_Z(z) = \dfrac{1}{\pi(1+z^2)}$.

Problem 6. The transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1,2,3,4\}$ is given below along with the matrix products P^2 and P^3.

$$P = \begin{pmatrix} 0.02 & 0.44 & 0.49 & 0.05 \\ 0.69 & 0.14 & 0.09 & 0.08 \\ 0.38 & 0.29 & 0.11 & 0.22 \\ 0.73 & 0.04 & 0.16 & 0.07 \end{pmatrix}$$
$$P^2 = \begin{pmatrix} 0.5267 & 0.2145 & 0.1113 & 0.1475 \\ 0.203 & 0.3525 & 0.3734 & 0.0711 \\ 0.4101 & 0.2485 & 0.2596 & 0.0818 \\ 0.1541 & 0.376 & 0.3901 & 0.0798 \end{pmatrix}$$
$$P^3 = \begin{pmatrix} 0.3085 & 0.3 & 0.3132 & 0.0783 \\ 0.4411 & 0.2498 & 0.1836 & 0.1255 \\ 0.338 & 0.2938 & 0.265 & 0.1032 \\ 0.469 & 0.2368 & 0.165 & 0.1292 \end{pmatrix}$$

(a) (4%) What is the numerical value of $P(X_4 = 1 \mid X_3 = 2, X_2 = 4, X_1 = 3, X_0 = 4)$?

See page 4 of notes13_markov_chains.pdf.

The answer is: $P(X_4 = 1 \mid X_3 = 2, X_2 = 4, X_1 = 3, X_0 = 4) = p_{21} = 0.69$.

(b) (4%) Let P_4 denote the probability function of the Markov chain started in state 4, that is, when the initial distribution is $a = (0,0,0,1)$.

Write $P_4(X_3 = 2, X_6 = 3, X_9 = 4)$ as a product of some of the numerical values given above. (**Write out all the factors explicitly, but do NOT evaluate the product!**)

See page 13 of notes13_markov_chains.pdf. The answer is: $p_{42}^{(3)} p_{23}^{(3)} p_{34}^{(3)} = 0.2368 \times 0.1836 \times 0.1032$