TEST #1
STA 5326 **Name:**____________________________________
September 26, 2022

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- You must show and explain your work (including your calculations) for all the problems (except the last one). **No credit is given without work or explanation!** This even includes the counting problems. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available (except for the last problem). If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- The different problems are not related. The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **11** pages. There are a total of **100** points.

General Remarks on grading:

I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).

I usually do not deduct points for copying errors (unless they make the answer ridiculous).

I do deduct points for algebra and calculus errors, but as long as the student's solution is basically correct (except for the algebra or calculus mistakes), I try to give them at least half the credit.

However, if an algebra or calculus mistake is bad enough to transform the problem so that the solution after the point of the mistake no longer resembles the correct solution, then the student only gets credit for the work up to the point of the mistake.

In general, if a student makes a mistake which completely changes the problem, and then the student correctly solves the changed problem, they only get credit up to the point of the mistake; they don't get credit for correctly solving the wrong problem.

Problem 1. A player is throwing darts at a circular target of radius **3**. Each of the player's darts lands **uniformly at random** on the target. The target has two circles marked on it. Circle A consists of all points at distance **1** from the center, and circle B consists of all points at distance **2** from the center.

See pages 14 and 20 of notes2.pdf

(a) (3%) What is the probability that a dart lands **inside** circle A?

(b) (3%) What is the probability a dart is inside circle A **given** that it is inside circle B?

(c) (6%) **Suppose 12 darts are thrown.** What is the probability at least two of them land inside A?

Similar to exercise 1.36.

If a student got a wrong answer to part (a) and used this wrong answer in answering parts (c) and (d) but otherwise did these parts correctly, give them full credit for parts (c) and (d).

(d) (5%) What is the probability at least two of them land inside **A given** at least one of them lands inside **A**?

Problem 2. Suppose the random variables $Y_1, Y_2, Y_3, \ldots$ are independent, and

$$Y_i \sim \text{Normal}(\mu = 3, \sigma^2 = 5)$$

for all i. Define

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{3^i} = \frac{Y_1}{3} + \frac{Y_2}{9} + \frac{Y_3}{27} + \cdots .$$

Calculate the following.

Similar to exercise A3

(a) (6%) EX

(b) (6%) $\text{Var}(X)$.

Problem 3. Define the function $f(x)$ by

$$f(x) = \begin{cases} 2e^{3x} & \text{if } x < 0 \\ 2e^{-6x} & \text{if } x \geq 0\,. \end{cases}$$

Similar to exercise 2.4.

(a) (6%) If X is a random variable with density (pdf) given by $f(x)$, find $P(X < t)$ for all t. Evaluate all integrals.

(b) (6%) Find $P(|X| < t)$ for all t. Evaluate all integrals.

Problem 4. (12%) Suppose X has density (pdf)

$$f_X(x) = \frac{3\sqrt{x}}{2\pi^3} \quad \text{for } 0 < x < \pi^2$$

and $Y = \sin\left(\sqrt{X}\right)$. Find the density (pdf) of Y.

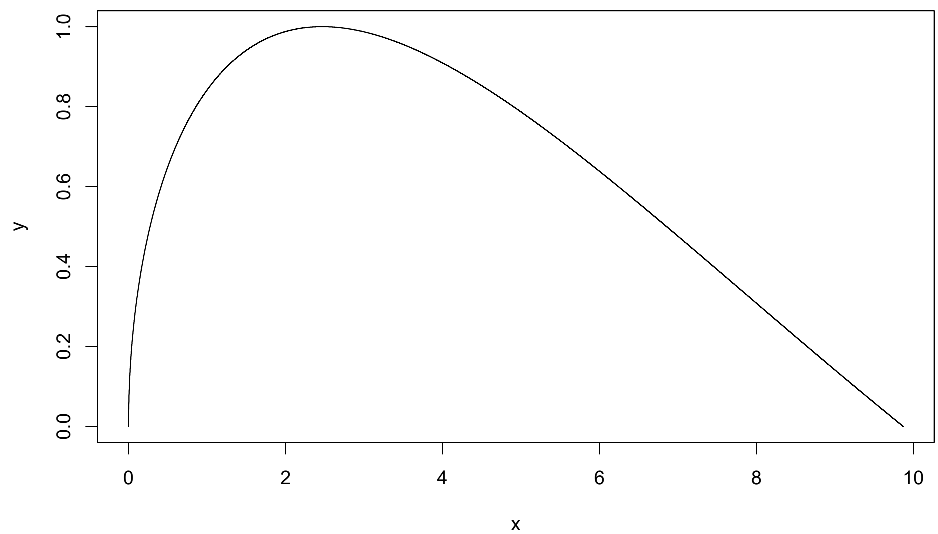
Helpful facts you may use without proof:

(1) The plot of $y = \sin(\sqrt{x})$ for $0 < x < \pi^2$ is given below.

(2) For $0 < y < 1$, solving $y = \sin(\sqrt{x})$ for x yields two solutions:

$$x = (\sin^{-1} y)^2 \quad \text{and} \quad x = (\pi - \sin^{-1} y)^2 \,.$$

(3) $\dfrac{d}{dy} \sin^{-1} y = \dfrac{1}{\sqrt{1-y^2}}$



This problem is similar to (but somewhat more complicated than) the example in notes3.pdf on pages 22, 23, and 25. The answer can be obtained using Theorem 2.1.8 (or the very similar formula on page 26 of notes3.pdf) or by the formula on page 24 of notes3.pdf.
The answer is

$$f_Y(y) = \frac{3}{\pi^3\sqrt{1-y^2}}\left((\sin^{-1} y)^2 + (\pi - \sin^{-1} y)^2\right) \quad \textit{for } 0 < y < 1 \,.$$

This is the same as the answer in the example on pages 22, 23, and 25 of notes3.pdf. (This is not a coincidence.)

[**Problem 4 continued**]

Additional Room For Problem 4 (if needed)

Problem 5. Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

(a) (6%) Describe (in general) what must be done to show that four events A, B, C, D are mutually independent.

This is exercise B8(a).

(b) (9%) Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

This is exercise B8(b).

Problem 6. (12%) Suppose 32 balls are placed at random into 11 cells. (The balls are placed independently of each other, with all cells being equally likely.) What is the probability there is at least one ball in every cell?

(Express your answer as a summation. Do NOT compute the sum!)

This is similar to exercise 1.20 which is equivalent to finding the probability of at least one ball in every cell when there are 12 balls in 7 cells. But the number of balls (32) and cells (11) in this problem is too big to use the solution approach on page 4 of solutions1_text.pdf. You should instead use the inclusion-exclusion approach on page 5 of solutions1_text.pdf.
Following this approach, the answer is:

$$1 - \sum_{i=1}^{11} (-1)^{i+1} \binom{11}{i} \left(\frac{11 - i}{11} \right)^{32} = 0.5624377\,.$$

Problem 7. (8%) The random variable Y has distribution function (cdf) given by

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 4\,, \\ \frac{4}{3}\left(1 - \frac{16}{y^2}\right) & \text{if } 4 \le y < 8\,, \\ 1 & \text{if } y \ge 8\,. \end{cases}$$

Find the distribution function (cdf) of $Z = 3(Y - 2)$.

This is similar to exercise 1.53.
Let $Z = g(Y)$ where $g(y) = 3(y-2)$. Since g is strictly increasing and continuous, we know (from page 9 of notes3.pdf) that

$$\begin{aligned} F_Z(z) &= F_Y(g^{-1}(z)) = F_Y((z+6)/3) \\ &= \begin{cases} 0 & \textit{if } g^{-1}(z) < 4\,, \\ \frac{4}{3}\left(1 - \frac{16}{(g^{-1}(z))^2}\right) & \textit{if } 4 \le g^{-1}(z) < 8\,, \\ 1 & \textit{if } g^{-1}(z) \ge 8\,. \end{cases} \\ &= \begin{cases} 0 & \textit{if } z < g(4)\,, \\ \frac{4}{3}\left(1 - \frac{16}{((z+6)/3)^2}\right) & \textit{if } g(4) \le z < g(8)\,, \\ 1 & \textit{if } z \ge g(8)\,. \end{cases} \\ &= \begin{cases} 0 & \textit{if } z < 6\,, \\ \frac{4}{3}\left(1 - \frac{144}{(z+6)^2}\right) & \textit{if } 6 \le z < 18\,, \\ 1 & \textit{if } z \ge 18\,. \end{cases} \end{aligned}$$

No work is required in the next problem. You will receive full credit just for stating the correct answers.

Problem 8. For events A and B, find formulas for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$, and $P(A \cap B)$. Put your answers in the blanks provided.

This problem combines exercises 1.2(abc) and 1.4.

(a) (3%) Either A or B but not both ________________________

Answer: $P(A) + P(B) - 2P(A \cap B)$

(b) (3%) At most one of A or B ________________________

Answer: $1 - P(A \cap B)$

(c) (3%) $(B \cap A) \cup (B \cap A^c)$ ________________________

Answer: $P(B)$

(d) (3%) $A \cup (B \cap A^c)$ ________________________

Answer: $P(A) + P(B) - P(A \cap B)$