TEST #2
STA 5326 **Name:**____________________________________
October 31, 2022

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- You must show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available (except on the last page). If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a moment generating function is undefined outside of some range, this range should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **12** pages. There are a total of **100** points.

Problem 1. Suppose X has density

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

where $\lambda > 0$ is a constant. Define $Y = X^\beta$ where $\beta > 0$ is a constant.

Similar to exercise 3.24(a).

(a) (8%) Find the density of Y.

[**Problem 1 continued**]

(b) (8%) Find the mean of Y.

Use the density you found in the previous part or use the "Law of the Unconscious Statistician" (which is easier).

Problem 2. (14%) Suppose that clicks on a Geiger counter occur according to a Poisson process with an averate rate of 2.0 clicks per second. What is the probability of the event that there are **exactly 4** clicks during the time interval $(3.5, 6.5)$ and **at least 2** clicks during the time interval $(8.0, 9.5)$?

Similar to the example on page 33 of notes6.pdf.

Problem 3. (14%) Use an argument involving indicator random variables to give an expression for $P(A \cap B^c \cap C^c)$. This expression should only contain terms chosen from the list

$$P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(B \cap C), P(A \cap B \cap C).$$

This is the lecture example on page 12 of notes4.pdf.

Problem 4. Suppose X is a discrete random variable with probability mass function (pmf) given by

$$f(x) = \begin{cases} \frac{1}{2}p(1-p)^{|x|-1} & \text{for } x = \pm 1, \pm 2, \pm 3, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < p < 1$, that is, $f(x) = \frac{1}{2}p(1-p)^{|x|-1}$ for all integers except 0 for which we have $f(0) = 0$.

This is most similar to the lecture notes on pages 7 to 10 of notes6.pdf. But it is also somewhat similar to the parts of exercise 2.33.

(a) (8%) Find the moment generating function (mgf) of X.

Calculating the mgf requires adding two summations, both similar to the summation you do for the mgf of the Geometric distribution, leading to $M_X(t) = \frac{1}{2}(M(t) + M(-t))$ *where* $M(t)$ *is the Geometric mgf. This leads to the answer*

$$M_X(t) = \frac{1}{2}\left(\frac{pe^{-t}}{1-(1-p)e^{-t}} + \frac{pe^{t}}{1-(1-p)e^{t}}\right) \quad \textit{for } \log(1-p) < t < -\log(1-p)\,.$$

[**Problem 4 continued**]

(b) (6%) Find the mean of X.

You can compute the mean from the mgf. Alternatively, since the pmf is symmetric about zero and we know the mean exists (since the mgf is finite in a neighborhood of zero), it follows that $EX = 0$.

[Problem 4 continued]

(c) (6%) Find the variance of X.

You can find the variance by using the mgf to find the second moment. Here is an alternative argument: The pmf is symmetric about zero. The positive side of the pmf is just $1/2$ *times the pmf of the Geometric(p) distribution, and the negative side of the pmf is just the reflection of this. Thus* $|X| \sim$ *Geometric(p). Let* $Y = |X|$. *Then*

$$\begin{aligned} Var(X) &= EX^2 - (EX)^2 = EX^2 - 0 = EX^2 = E|X|^2 \\ &= EY^2 = Var(Y) + (EY)^2 = \frac{1-p}{p^2} + \left(\frac{1}{p}\right)^2 = \frac{2-p}{p^2}. \end{aligned}$$

Problem 5. (14%) Suppose that $EX^2 < \infty$. Show that $E(X-a)^2$ is minimized over a when $a = EX$. Make sure to show that $a = EX$ is indeed a minimum. (If you wish, you may assume that X has a density f_X, although this is not really needed.)

This is exercise 2.19.

Problem 6. An urn contains n balls. One of them is red, the others are all green. The balls are drawn out one by one withOUT replacement. Let X be the number of draws needed to get the red ball.

(a) (10%) Find EX.

This problem is the same as exercise 3.4 but with a more boring story. The answer is $\frac{n+1}{2}$.

[**Problem 6 continued**]

(b) (4%) Now suppose there are two red balls in the urn, and the other $n-2$ balls are green. The balls are drawn out one by one withOUT replacement. Let X be the number of draws needed to get the first red ball. What is EX?

The pmf of X *is* $P(X=i) = \frac{\binom{n-i}{1}}{\binom{n}{2}} = \frac{\binom{2}{1}\binom{n-2}{i-1}(i-1)!}{\binom{n}{i}i!} = \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdots\frac{n-i}{n-i+2}\cdot\frac{2}{n-i+1}$

The first two expressions come from different counting arguments, and the third expression from an urn-style argument. After finding the pmf, then $EX = \sum_{i=1}^{n} i\cdot P(X=i) = \frac{2}{n(n-1)}\sum_{i=1}^{n} i\cdot(n-i) = \frac{n+1}{3}$. *This answer can also be obtained by a heuristic argument. Let* U, V, W *denote the number of green balls drawn before the first red ball, the number drawn between the two red balls, and the number drawn after the second red ball, respectively. We know that* $U+V+W = n-2$ *and it is intuitive (or is it?) that* $EU = EV = EW = \frac{n-2}{3}$. *Since* $X = U+1$ *this implies* $EX = 1 + \frac{n-2}{3} = \frac{n+1}{3}$.

No work is required for the problems on this page. You will receive full credit for stating the correct answer. Put your answer in the blank provided.

Problem 7. (4%) Let $X \sim \text{Gamma}(\alpha, \beta = 1)$ and define

$$\psi(\alpha) = P(\alpha - 0.5 < X < \alpha + 0.5)\,.$$

What is the value of

$$\lim_{\alpha \to \infty} \psi(\alpha)\,?$$

The answer is 0. This follows from the Central Limit Theorem (CLT) or, equivalently, from the Normal approximation to the Gamma distribution for large α.

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Problem 8. (4%) Let $X_1, X_2, X_3, \ldots$ be iid $\text{Poisson}(\lambda = 1)$ and define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. What is the value of

$$\lim_{n \to \infty} P(0.5 < \bar{X}_n < 1.5)\,?$$

The answer is 1. This follows from the Law of Large Numbers (LLN).

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