TEST #1
STA 5326 **Name:**____________________________________
October 2, 2023

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc). Calculators are NOT allowed.
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- You must show and explain your work (including your calculations) for all the problems (except for problems labeled **NWR**). **No credit is given without work or explanation!**. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available (except for problems labeled **NWR**). If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- **No work is required** for the problems marked **NWR**. For these problems, you will receive full credit just for stating the correct answer.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and pages. There are a total of **100** points.

Problem 1. n figure skaters perform in a random order. Each skater is given a rating by a panel of judges. Assume there are no ties among the skaters, that is, there are no skaters that are rated the same.

(a) (12%) If the i-th skater is the best so far, what is the probability this skater will be the best overall?

This exercise is 1.32 with a slightly changed story. Two solutions (a counting solution and a probability solution) are given in solutions1_text.pdf on pages 17–20. There may be other solutions.

[**Problem 1 continued**]

(b) (12%) Suppose that prizes are given to the top three skaters. If the i-th skater is the best so far, what is the probability this skater does NOT receive a prize? (Assume $n > 3$ and $1 \le i \le n-3$.)

This is similar to exercise 1.32.

Solution #1: *(A 'counting' solution) The sample space Ω consists of all the $n!$ permutations $\omega = (r_1, r_2, \ldots, r_n)$ of the numbers $1, 2, \ldots, n$. Here r_i represents the final ranking of the i-th skater (where 1 = worst and n = best). The random ordering of the skaters implies all the permutations are equally likely. Let $A = \{$i-th skater is the best so far$\}$, and $B = \{$ i-th skater does NOT win a prize$\}$. We are interested in $P(B \mid A) = P(A \cap B)/P(A) = \#(A \cap B)/\#(A)$. Just like in the posted solution of 1.32 we have $\#(A) = \binom{n}{i}(i-1)!(n-i)!$. If the best among the first i skaters does NOT win a prize, then it must be that the random ordering puts all of the top 3 skaters among the last $n-i$ skaters. A permutation ω of skaters in $A \cap B$ is such that the i-th skater is the best so far but does NOT win a prize. This may be constructed in three steps: (1) Choose i skaters from among the lowest-ranked $n-3$ to be the first i skaters; this may be done in $\binom{n-3}{i}$ ways. (2) Choose the best of these for position i and assign the other $i-1$ to the first $i-1$ positions; this may be done in $(i-1)!$ ways. (3) Order the remaining $n-i$ skaters (which include the top 3 skaters) in the remaining $n-i$ positions; this can be done in $(n-i)!$ ways. Taking the product gives $\#(A \cap B) = \binom{n-3}{i}(i-1)!(n-i)!$ ways. Dividing this by $\#(A)$ given earlier and simplifying gives $P(B \mid A) = \binom{n-3}{i} \Big/ \binom{n}{i} = \dfrac{(n-i)(n-i-1)(n-i-2)}{n(n-1)(n-2)}$.*

Solution #2: *(A 'probability' solution.) This is somewhat heuristic. The best skater among the first i will NOT get a prize only if none of the top 3 skaters are among the first i skaters. Call this event D. The top 3 skaters are equally likely to be in any of the 3 positions in the random ordering of the skaters. Let C_j be the event that the top j-th skater in the rankings is NOT among the first i in the order of skating. Then $D = C_1 \cap C_2 \cap C_3$ and $P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2|C_1)P(C_3|C_1 \cap C_2) = \dfrac{(n-i)}{n} \cdot \dfrac{(n-i-1)}{(n-1)} \cdot \dfrac{(n-i-2)}{(n-2)}$ which is equal to the answer given earlier.*

Solution #3: *(A 'counting' solution; a variant of Solution #2.) As noted above, the best skater among the first i will NOT get a prize only if none of the top 3 skaters are among the first i skaters. This is the same as saying the top 3 skaters are among the last $n-i$ skaters. Call this event D. Whether the event D occurs depends only on the positions of the top 3 skaters in the ordering. So (intuitively, at least) we can reduce our sample space to the $n(n-1)(n-2)$ possible choices of the positions of the top 3 skaters, all of which are equally likely. In this reduced sampling space, the number of possibilities in D, i.e., for which the top 3 skaters are all among the last $n-i$ skaters, is $(n-i)(n-i-1)(n-i-2)$. Thus $P(D) = \#(D)/\#(\Omega) = \dfrac{(n-i)(n-i-1)(n-i-2)}{n(n-1)(n-2)} = \binom{n-i}{3} \Big/ \binom{n}{3}$.*

Comment on Solutions #2 and #3: *Let A, B, D be the events defined in the earlier solutions. Both Solutions #2 and #3 rely on $P(B|A) = P(D)$. But the definition of D says nothing about skater i being the best so far, so why should this be true? (This is the heuristic aspect in Solutions*

#2 and #3.) The reason is that $A \cap B = A \cap D$ *so that*

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \cap D)}{P(A)} = \frac{P(D)P(A|D)}{P(A)} = P(D)$$

because $P(A|D) = P(A) = 1/i$ *; the random ordering of the skaters guarantees that all of the first* i *skaters are equally likely to be the best among the first* i *even if none of the top 3 skaters are among the first* i *skaters.*

Solution #4: *(A 'probability' solution.) As in the earlier solutions, let* $A = \{\text{skater } i \text{ is the best so far}\}$ *and* $B = \{\text{skater } i \text{ does NOT win a prize}\}$*. Then* $B^c = F_1 \cup F_2 \cup F_3$ *where* $F_j = \{\text{skater } i \text{ is in } j\text{-th place overall}\}$ *so that*

$$\begin{aligned} P(B|A) &= 1 - P(B^c|A) = 1 - P(F_1|A) - P(F_2|A) - P(F_3|A) \\ &= 1 - \frac{P(F_1 \cap A)}{P(A)} - \frac{P(F_2 \cap A)}{P(A)} - \frac{P(F_3 \cap A)}{P(A)} \\ &= 1 - \frac{i}{n} - \frac{i(n-i)}{n(n-1)} - \frac{i(n-i)(n-i-1)}{n(n-1)(n-2)} \end{aligned}$$

which (after a little algebra) can be shown to agree with the earlier answer. The final answer is justified as follows. We know from the solution of exercise 1.32 that $F_1 \cap A = F_1$, $P(F_1) = 1/n$, *and* $P(A) = 1/i$ *so that* $P(F_1|A) = i/n$*. The event* $F_2 \cap A$ *occurs if the 2nd place skater is in position* i *and the 1st place skater is among the last* $n - i$ *skaters; this has probability* $\frac{1}{n} \cdot \frac{n-i}{n-1}$ *so that* $P(F_2|A) = \frac{i \cdot (n-i)}{n \cdot (n-1)}$*. Finally,* $F_3 \cap A$ *occurs if the 3rd place skater is in position* i *and the 1st and 2nd place skaters are among the last* $n-i$ *skaters; this has probability* $\frac{1}{n} \cdot \frac{n-i}{n-1} \cdot \frac{n-i-1}{n-2}$ *so that* $P(F_3|A) = \frac{i(n-i)(n-i-1)}{n(n-1)(n-2)}$.

Solution #5: *This solution is a variant of Solution #4 and is only for those few among you who might be familiar from other courses with facts about order statistics. Suppose that the skater's ratings* $X_1, X_2, \ldots, X_n$ *are iid random variables with density* g *and cdf* G*. This guarantees that the* $n!$ *possible rankings of the skaters are equally likely and that the rankings are independent of the order statistics of the ratings. We use the events* A, B, F_1, F_2, F_3 *from Solution #4 and the fact that*

$$P(B|A) = 1 - P(F_1|A) - P(F_2|A) - P(F_3|A)\,. \tag{1}$$

Condition on X_i *and* A*, that is, suppose we know the rating* X_i *of skater* i *and also that skater* i *is the best among the first* i *skaters. Then* F_j *occurs (i.e., skater* i *is ranked* j*-th overall) only if exactly* $j-1$ *skaters among the last* $n-i$ *skaters have ratings which exceed* X_i*. This has probability* $P(F_j|X_i, A) = \binom{n-i}{j-1} G(X_i)^{n+1-i-j}(1 - G(X_i))^{j-1}$*. We can now obtain* $P(F_j|A)$ *by integrating* $P(F_j|X_i = x, A)$ *times the density of* $X_i|A$*; this is the density of the maximum of* i

rv's which are iid with density g which is equal to $i\,G(x)^{i-1}g(x)$. This leads to

$$
\begin{aligned}
P(F_j|A) &= \int_{-\infty}^{\infty} \binom{n-i}{j-1} G(x)^{n+1-i-j}(1-G(x))^{j-1} \cdot i\,G(x)^{i-1}g(x)\,dx \\
&= i\binom{n-i}{j-1}\int_{-\infty}^{\infty} G(x)^{n-j}(1-G(x))^{j-1}g(x)\,dx \\
&= \frac{i\binom{n-i}{j-1}}{n\binom{n-1}{n-j}}\int_{-\infty}^{\infty} n\binom{n-1}{n-j}G(x)^{n-j}(1-G(x))^{j-1}g(x)\,dx \\
&= \frac{i\binom{n-i}{j-1}}{n\binom{n-1}{n-j}} = \frac{i}{n}\cdot\frac{(n-j)!}{(n-1)!}\cdot\frac{(n-i)!}{(n-i-j+1)!}
\end{aligned}
$$

which agrees with $P(F_j|A)$ given in Solution #4 for $j = 1, 2, 3$. Thus, plugging these values into (1) gives than same final answer as Solution #4. In one step above, we evaluated an integral by juggling the constants so that the integrand was exactly the known density of an order statistc (which therefore integrated to one). Obviously, Solution #5 involves much more work than Solution #4, but they do lead to the same answer, which is reassuring.

Problem 2. Suppose n people play Russian roulette. Each person has a gun which fires with probability π when the trigger is pulled. (Assume the guns are independent of each other and successive shots of the same gun are independent.) A round of play consists of every one who is still alive raising the guns to their temples and firing simultaneously. Play continues until everyone is dead.

This situation is the same as that in Exercise B3.

(a) (12%) What is the probability that one or more people are still alive after k rounds of play?

This exercise is exactly the same as B3(a). There is a posted solution in solutions1_ABexercises.pdf and another in B3a_alternate_solution.pdf.

[**Problem 2 continued**]

(b) (12%) The last person (or persons) to die receives a prize (flowers on the grave). What is the probability this prize goes to **exactly two persons**? Assume $n > 2$. (Note: The answer may not have a simple form, so do not worry if your answer is messy or contains summations you do not know how to do.)

This exercise is a modified version of B3(b). The solution is very similar. Let A be the event that the prize goes to exactly two persons. Then $A = \bigcup_{i,j} A_{i,j}$ where $A_{i,j} = \{$only persons i and j get the prize$\}$. These events are disjoint and any two people have the same probability of getting the prize so that $P(A) = \binom{n}{2} P(A_{1,2})$. Let B_k, $k \geq 2$, be the events that persons 1 and 2 get the prize upon the conclusion of round k. Clearly, these events are disjoint so that $P(A_{1,2}) = \sum_{k=2}^{\infty} P(B_k)$. Since

$$B_k = \{\textit{Persons 1, 2 die in round } k\} \cap \{\textit{Persons } 3, 4, \ldots, n \textit{ die before round } k\}$$

and the persons are independent, we have

$$P(B_k) = \left[(1-\pi)^{k-1}\pi\right]^2 \left[1 - (1-\pi)^{k-1}\right]^{n-2} .$$

To justify the right factor above we use arguments like those used in part (a) above. Putting this all together gives the final answer:

$$P(A) = \binom{n}{2} \sum_{k=2}^{\infty} \left[(1-\pi)^{k-1}\pi\right]^2 \left[1 - (1-\pi)^{k-1}\right]^{n-2} .$$

Problem 3. In each of the following find the density (pdf) of Y.

(a) (12%) $Y = X^4$ and X has density $f_X(x) = 2(x-1)$ for $1 < x < 2$.

This problem is similar to the parts of Exercise 2.1. The answer is

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right| \quad \textit{for } y \in \mathcal{Y} \\ &= 2(y^{1/4}-1)\left|\frac{d}{dy}\,y^{1/4}\right| \quad \textit{for } y \in (1^4, 2^4) \\ &= 2(y^{1/4}-1)\cdot\frac{1}{4}y^{-3/4} \quad \textit{for } 1 < y < 16\,. \end{aligned}$$

It is important in the above to state that $1 < y < 16$.

[**Problem 3 continued**]

(b) (12%) $Y = X^4$ and X has density $f_X(x) = (2x+7)/30$ for $-3 < x < 2$.

This problem is similar to exercise 2.7. Note that $g(x) = x^4$ is not monotonic in $(-3,2)$, but we can break $(-3,2)$ into the intervals $(-3,0)$ and $(0,2)$ in which it is monotonic. Using the notation from page 26 of notes3.pdf, the answer is

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^{2} f_X\left(g_i^{-1}(y)\right) \left| \frac{d}{dy}\, g_i^{-1}(y) \right| I_{B_i}(y) \\
&= \frac{(2(-y^{1/4})+7)}{30} \left| \frac{d}{dy}(-y^{1/4}) \right| I_{(0,(-3)^4)}(y) + \frac{(2y^{1/4}+7)}{30} \left| \frac{d}{dy}\, y^{1/4} \right| I_{(0,2^4)}(y) \\
&= \frac{(2(-y^{1/4})+7)}{30} \cdot \frac{1}{4} y^{-3/4} I_{(0,81)}(y) + \frac{(2y^{1/4}+7)}{30} \cdot \frac{1}{4} y^{-3/4} I_{(0,16)}(y) \\
&\quad \textit{(The above answer is good enough for full credit.)} \\
&= \frac{(-2y^{1/4}+7)}{30} \cdot \frac{1}{4} y^{-3/4} I_{(16,81)}(y) + \frac{14}{30} \cdot \frac{1}{4} y^{-3/4} I_{(0,16)}(y) \\
&= \begin{cases} \dfrac{(-2y^{1/4}+7)}{120} \cdot y^{-3/4} & \textit{for } 16 < y < 81 \\ \dfrac{7}{60}\, y^{-3/4} & \textit{for } 0 < y < 16 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}
$$

(The endpoints of the intervals can be handled differently and the "otherwise" clause can be omitted).

Problem 4. A monomial of degree d in the k variables $x_1, x_2, \ldots, x_k$ is a product of the form $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ where the k exponents $i_1, i_2, \ldots, i_k$ are nonnegative integers which sum to d. (Note that zero is allowed as an exponent.)

Some monomials of degree 6 in the 3 variables x, y, z are listed here as examples:

(1) $x^6y^0z^0 = x^6$	(2) $x^0y^0z^6 = z^6$	(3) $x^4y^2z^0 = x^4y^2$
(4) $x^2y^4z^0 = x^2y^4$	(5) $x^2y^1z^3 = x^2yz^3$	(6) $x^1y^3z^2 = xy^3z^2$

All six of these examples are different monomials because they are different functions of x, y, z. Rearranging the order of factors does not change their product; xy^2z^3, y^2z^3x, and z^3xy^2 are all considered to be the same monomial because they are equal.

(a) (12%) How many different monomials of degree d in the k variables $x_1, x_2, \ldots, x_k$ are there?

This is similar to Exercise 1.19 (counting partial derivatives). By their definition the monomials of order d in k variables are in one-to-one correspondence with the k-tuples of nonnegative integers which sum to d which are in one-to-one correspondence with the possible arrangements of d counters and $k-1$ markers (by the argument given in the long solution to Exercise 1.19). Therefore the number of monomials is $\binom{d+k-1}{k-1} = \binom{d+k-1}{d}$.

Some students may give this answer and refer to "Unordered, with replacement" method of counting as described in the text. This should get full credit also.

(b) (3%) How many **different** polynomials are there which are sums of two **different** monomials of degree d in the k variables $x_1, x_2, \ldots, x_k$? **(NWR)**

(Example: $x^4y^2 + xy^3z^2$ is a sum of two different monomials of degree 6 in the 3 variables x, y, z. Changing the order of the two terms does not alter the sum so that both $x^4y^2 + xy^3z^2$ and $xy^3z^2 + x^4y^2$ are considered to be the same polynomial.)

The number of different polynomials which are sums of two different monomials is just the number of ways to choose two monomials from the entire set of monomials (of degree d in k variables). This is $\binom{\binom{d+k-1}{d}}{2}$.

If students give a wrong answer to the previous part (say, W) but then give $\binom{W}{2}$ as the answer to this part, then give them full credit for this part.

Problem 5. (4%) Approximately one-third of all human twins are identical (one-egg) and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, one-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth.

What is the probability that a U.S. birth results in fraternal twins, with one being male and the other female?

The answer is:
$P(\textit{twin birth})P(\textit{fraternal} \mid \textit{twin birth})P(\textit{boy and girl} \mid \textit{fraternal twins}) = \frac{1}{90} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{270}.$
Some work should be shown.

In the remaining questions, **circle** the **single** correct response. **(NWR)**

Problem 6. (3%) If A, B, and C are mutually **exclusive** events, all having positive probability, then $P(A \cup B \mid B \cup C) =$ ________.

a) 0

b) $P(A)$

c)$\star$ $\dfrac{P(B)}{P(B)+P(C)}$

d) $\dfrac{1}{P(C)}$

e) $\dfrac{P(B)+P(A)P(C)-P(A)P(B)P(C)}{P(B)+P(C)-P(B)P(C)}$

f) $\dfrac{P(A)P(B)P(C)}{P(B)+P(C)}$

g) $\dfrac{P(A)P(B)+P(B)P(C)-P(A)P(B)P(C)}{P(B)+P(C)-P(B)P(C)}$

h) $\dfrac{P(B)}{P(B)+P(C)-P(B)P(C)}$

Problem 7. (3%) A random variable that is continuous but **not** absolutely continuous ________.

a) has a **p**df which is continuous except at finitely many points

b) has a **c**df which is continuous except at finitely many points

c) has a continuous **c**df with finitely many flat intervals

d) has a continuous **c**df and a **p**df which is NOT continuous

e)$\star$ has a continuous **c**df but does NOT have a **p**df

f) has a **c**df with NO jumps but with finitely many points where the derivative does NOT exist

Problem 8. (3%) Which one of the following statements is always true?

a) $P(A \cap B^c \cap C^c) \geq P(A) - P(A \cap B) + P(A \cap B \cap C)$

b)$\star$ $P((A \cap B) \cup C) \leq P(A \cap B) + P(C)$

c) $1 - P(A^c \cap B^c) \geq P(A) + P(B)$

d) $1 - P(A^c \cup B^c) = P(A)P(B)$

e) $P(A \cap B) \geq P(A)P(B)$