TEST #2
STA 5326 **Name:**_____
November 6, 2023

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use what you need to write with (pens, pencils, erasers, etc) and an ordinary scientific calculator.
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Your calculator should NOT be able to do algebra or calculus. It should have a small screen and limited memory. It should NOT have any internet or phone capability. Computers or tablets are NOT allowed.
- You must show and explain your work for all the problems (except for the last problem labeled **NWR**). **No credit is given without work or explanation!**. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available (except for problem labeled **NWR**). If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- **No work is required** for the very last problem which is marked **NWR**. For this problem, you will receive full credit just for stating the correct answer.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely unless specifically requested. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **9** pages. There are a total of **100** points.

Problem 1. Suppose X has the probability mass function (pmf)

$$f(x) = (1-p)^2(x+1)p^x \quad \text{for } x = 0, 1, 2, 3, \ldots .$$

where $0 < p < 1$.

In the parts (a) and (b) below, **do NOT use any results from the Appendix** (the Table of Distributions). You may use without proof the fact that the formula above is a valid pmf for all p satisfying $0 < p < 1$.

(a) (10%) Show that the moment generating function (mgf) of X is

$$M(t) = \frac{(1-p)^2}{(1-pe^t)^2} .$$

(b) (3%) What is the range of values of t for which the mgf of X is finite (well-defined) and given by the above formula?

[**Problem 1 continued**]

In the next part you are allowed to use the Appendix.

(c) (10%) Define a new random variable by $Y = (1-p)X$. Use mgf's to show that, as $p \uparrow 1$, the random variable Y converges in distribution to a random variable with a Gamma distribution. Specify the values of the parameters α and β for this Gamma distribution.

Note: $p \uparrow 1$ means the same as $p \to 1-$.

Problem 2. Suppose that X has density given by

$$f(x) = \frac{7^8}{x^8} \quad \text{for } x > 7\,.$$

(a) (10%) Find EX^k for all positive integers k. (More precisely, carefully state the values of k for which EX^k is well-defined and finite, and give the value of EX^k for these values of k.)

(b) (6%) What is the value of $P(X > 11 \,|\, X > 9)$?

Problem 3. In the fictional slightly red state of Arozida, 50.5% of the registered voters plan to vote Republican in the next election, 49.0% plan to vote Democrat, and 0.5% plan to vote Libertarian. A random sample of 400 registered voters is selected from this state.

In **both** parts (a) and (b) below:
use an appropriate approximation and **give a decimal answer**.

(a) (12%) What is the approximate probability the sample contains at least 201 people who plan to vote **Democrat** in the next election?

(b) (6%) What is the approximate probability the sample contains exactly 3 people who plan to vote **Libertarian**?

Problem 4. (11%) Suppose X has density

$$f(x) = \frac{3x^5}{\beta^2} e^{-x^3/\beta} \quad \text{for } x > 0.$$

Find EX^2.

Problem 5. (11%) Use an argument involving indicator random variables to give an expression for

$$P(A^c \cap B \cap C^c) + P(A \cap B^c \cap C)\,.$$

This expression should only contain terms chosen from the list

$$1, P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(B \cap C), P(A \cap B \cap C)\,.$$

Problem 6. (8%) A nuclear power plant had a small radiation leak which escaped into the atmosphere and spread around the world, very slightly elevating the risk of death from cancer for everyone on Earth. Suppose that all of the 8.1 billion people on earth are placed in order 1, 2, 3, ..., 8.1×10^9 according to their degree of exposure to this radiation, and that person i in this ordering has probability $p_i = 10^{-6} \exp(-10^{-6}\, i)$ of dying from cancer due to this radiation leak.

What is the approximate probability that at least 2 people die of cancer due to this radiation?

Problem 7. (8%) A lightbulb has an exponentially distributed lifetime with a mean of 10,000 hours. If it has been in use for 13,268 hours and is still working, what is the approximate probability it will fail during the next hour of use? (Give a decimal answer.)

Problem 8. (5%) **(NWR)** An urn contains 7 red balls and 13 green balls. The balls are randomly drawn from the urn one by one until the urn is empty. What is the probability that the last two balls drawn from the urn are both red?