TEST #3
STA 5326 **Name:**______________________________________
December 6, 2023

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use what you need to write with (pens, pencils, erasers, etc) but nothing else.
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Calculators are NOT allowed!
- You must show and explain your work for all the problems (except 6(b) and 8, the multiple choice problem). **No credit is given without work or explanation!**. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available. If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely unless specifically requested. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **9** pages. There are a total of **100** points.

Problem 1. Let X and Y have a joint density given by

$$f_{X,Y}(x,y) = 24\left(\frac{y}{x+y}\right)^2 e^{-(x+y)} \quad \text{for } 0 < y < x < \infty.$$

This joint density is non-zero in that part of the first quadrant which lies between the x-axis and the line $y = x$. Define the random variables $U = X + Y$ and $Z = \dfrac{Y}{X+Y}$.

(a) (12%) Find the joint density of U and Z. Make sure to specify the support of the joint density.

(b) (6%) Are U and Z independent? (Justify your answer.)

Problem 2. (8%) Suppose we have the following hierarchical (two-stage) model:

$$X \sim \text{Weibull}(\gamma = 4, \beta = 3)\,,$$
$$Y \mid X \sim \text{Normal}(\mu = 2e^X + \cos(X)\,,\, \sigma^2 = X^2)\,.$$

Find $E(e^{-X}(Y - \cos X))$.

Problem 3. Suppose (X, Y) has a joint density $f(x, y)$ given by

$$f(x,y) = \frac{1}{\pi\sqrt{2}} \exp\left[-2x^2 + \sqrt{2}\,x(y-3) - \frac{(y-3)^2}{2}\right] \quad \text{for } -\infty < x < \infty,\ -\infty < y < \infty$$

(a) (12%) Find $f_Y(y)$, the marginal density of Y. (To receive full credit, you should derive the marginal from the joint density by direct calculations.)

[**Problem 3 continued**]

(b) (10%) Find $f_{Y|X}(y|x)$, the conditional density of Y given $X = x$.

Problem 4. A random point (X, Y) is distributed uniformly on the region

$$A = \{(x, y) : x > 0, y > 0, x^2 + y^2 < 1\},$$

consisting of the points in the first quadrant which are inside the unit circle.

(a) (6%) Let $F(x, y)$ be the joint cdf (cumulative distribution function) of (X, Y). Evaluate $F(c, d)$ where $(c, d) \in A$ is an arbitrary point inside A.

(b) (6%) Find $P(X + Y > 1)$.

Problem 5. Let X and Y be independent random variables with densities

$$f_X(x) = \frac{1}{2}\,e^{-|x|} \quad \text{for} \quad -\infty < x < \infty \quad \text{and}$$
$$f_Y(y) = \frac{1}{\pi(1+y^2)} \quad \text{for} \quad -\infty < y < \infty\,.$$

Define a new random variable Z by

$$Z = \begin{cases} Y & \text{if } XY > 0 \\ -Y & \text{if } XY < 0\,. \end{cases}$$

(a) (10%) What is the density of Z? (Give a precise formula and prove your answer.)

(b) (5%) Are Z and X independent? (Answer YES or NO and prove your answer.)

Problem 6. Suppose X and Y are iid (independent and identically distributed) Poisson(λ) random variables.

(a) (10%) What is the value of the conditional probability $P(X = 2 \,|\, X + Y = 6)$? (Give a detailed argument justifying your answer.)

(b) (5%) Now suppose X, Y, Z are iid Poisson(λ) random variables. What is the value of the conditional probability $P(X = 2 \,|\, X + Y + Z = 6)$?

(Here you will receive full credit just for stating the correct answer. **No work is required. NWR**)

Problem 7. (6%) A bivariate transformation $s = x^2 + y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ has the inverse transformation

$$x = \sqrt{s}\cos\theta\,, \quad y = \sqrt{s}\sin\theta\,.$$

Find the Jacobian of this inverse transformation. (Show your work in detail.)

Problem 8. (4%) For **discrete** random variables X and Y, the quantity $f_{Y|X}(y|x)$ is the same as __________. (Circle the single correct response.)

a) $\sum_x P(Y = y, X = x)$ **b)** $\dfrac{f_{X,Y}(x,y)\,dx\,dy}{f_Y(y)\,dy}$ **c)** $f_{X,Y}(x,y)\,dx\,dy$ **d)** $\dfrac{P(X = x, Y = y)}{P(Y = y)}$

e) $P(Y = y \mid X = x)$ **f)** $\dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ **g)** $f_Y(y)f_{X|Y}(x|y)$ **h)** $f_X(x)f_{X|Y}(x|y)$