TEST #3
STA 5326 **Name:**____________________________________
December 6, 2023

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use what you need to write with (pens, pencils, erasers, etc) but nothing else.
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Calculators are NOT allowed!
- You must show and explain your work for all the problems (except 6(b) and 8, the multiple choice problem). **No credit is given without work or explanation!**. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Partial credit is available. If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely unless specifically requested. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **9** pages. There are a total of **100** points.

Problem 1. Let X and Y have a joint density given by

$$f_{X,Y}(x,y) = 24\left(\frac{y}{x+y}\right)^2 e^{-(x+y)} \quad \text{for } 0 < y < x < \infty.$$

This joint density is non-zero in that part of the first quadrant which lies between the x-axis and the line $y = x$. Define the random variables $U = X + Y$ and $Z = \dfrac{Y}{X+Y}$.

(a) (12%) Find the joint density of U and Z. Make sure to specify the support of the joint density.

This is similar to exercise 4.24; it involves the same bivariate transformation.

The joint density is

$$f_{U,Z}(u,z) = 24z^2 \cdot ue^{-u} \quad \textit{for } 0 < z < 1/2,\ 0 < u < \infty.$$

(b) (6%) Are U and Z independent? (Justify your answer.)

Yes, U and Z are independent. The joint density factors as a function of u times a function of z, and the support is a product set.

Independence can be argued in different ways (which are all equivalent, really). For example, students can find the marginal densities $f_U(u)$ and $f_Z(z)$ and show that the joint density $f_{U,Z}(u,z) = f_U(u)f_Z(z)$ for all $(u,z) \in \mathbb{R}^2$.

Problem 2. (8%) Suppose we have the following hierarchical (two-stage) model:

$$X \sim \text{Weibull}(\gamma = 4, \beta = 3)\,,$$
$$Y \mid X \sim \text{Normal}(\mu = 2e^X + \cos(X)\,,\, \sigma^2 = X^2)\,.$$

Find $E(e^{-X}(Y - \cos X))$.

This is similar to the example on page 7 of notes12.pdf and uses "iterated expectations" and the "Useful General Properties" on page 6 of notes12.pdf.

$$\begin{aligned}
E(e^{-X}(Y - \cos X)) &= EE(e^{-X}(Y - \cos X) \mid X) \\
&= E[e^{-X} E(Y - \cos X \mid X)] \\
&= E[e^{-X}\{E(Y \mid X) - \cos X\}] \\
&= E[e^{-X}\{2e^X + \cos X - \cos X\}] \\
&= E[e^{-X}\{2e^X\}] \\
&= E[e^{-X} \cdot 2e^X] = E[2] = 2
\end{aligned}$$

Problem 3. Suppose (X, Y) has a joint density $f(x, y)$ given by

$$f(x,y) = \frac{1}{\pi\sqrt{2}} \exp\left[-2x^2 + \sqrt{2}\,x(y-3) - \frac{(y-3)^2}{2}\right] \quad \text{for } -\infty < x < \infty,\ -\infty < y < \infty$$

This is similar to exercise 4.45.

(a) (12%) Find $f_Y(y)$, the marginal density of Y. (To receive full credit, you should derive the marginal from the joint density by direct calculations.)

This part is similar to 4.45(a).

$f(x, y)$ is a bivariate normal density, but it is **not** *necessary to remember the bivariate density to solve this problem. To get full credit, students must obtain the marginal $f_Y(y)$ by integrating the joint density over x. This is a bit tedious, but straightforward; it requires completing the square in x and using knowledge of the normal density to perform the integral over x. The answer is*

$$f_Y(y) = \frac{1}{2\sqrt{\pi}} e^{-(y-3)^2/4}, \quad -\infty < y < \infty\,.$$

This is the $N(\mu = 3, \sigma^2 = 2)$ density, but students are not required to state this.

Some students may have memorized the bivariate normal density, and they might recognize (by playing around a bit) that $f(x, y)$ is the density of the bivariate normal distribution with $\mu_X = 0, \sigma_X^2 = 1/2, \mu_Y = 3, \sigma_Y^2 = 2, \rho = 1/\sqrt{2}$. Then using well known properties of the bivariate normal, they know that $Y \sim N(3, 2)$. If they state $Y \sim N(3, 2)$ as their answer and use this argument for it, they should receive (maybe) 8 points (but NOT full credit, since they were told to use direct calculation to get full credit).

[**Problem 3 continued**]

(b) (10%) Find $f_{Y|X}(y|x)$, the conditional density of Y given $X = x$.

This is similar to 4.45(b).

Sketch of Solution:
$f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$, *so that, if we calculate* $f_X(x)$ *by integrating* $f(x,y)$ *over* y*, and then compute and simplify this ratio, we will have found* $f_{Y|X}$.

Students should recognize (using the "Handy Fact" on page 11 of notes12.pdf) that $f(x,y)$ *is a bivariate normal density and remember that the conditional will therefore be a (univariate) normal distribution. So students should write their final answer in the usual form for a normal density and lose (maybe) one point if their answer is not expressed in this form (but is in some algebraically equivalent form).*

The solution outlined above involves finding $f_X(x)$ *as one of its steps. But this can be avoided. The conditional* $f_{Y|X}(y|x)$ *may be regarded as a function of* y *in which* x *is kept fixed at some arbitrary value. Since integrating this function over* y *must yield one, we can always discard multiplicative constants without losing any information, since the missing constant can always be recovered in the end by requiring our final answer to integrate to one. (Here a "constant" is any quantity which does* **not** *involve* y*.) Since the marginal* $f_X(x)$ *does not involve* y*, it is a constant and can be discarded. To indicate when we have dropped a multiplicative constant we use the symbol* $\propto$ *instead of* $=$*. Here is a solution which uses this idea.*

Alternate Solution (fairly detailed):

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&\propto f(x,y) \\
&\propto \exp\left[-2x^2 + \sqrt{2}\,x(y-3) - \frac{(y-3)^2}{2}\right] \\
&\propto \exp\left[\sqrt{2}\,x(y-3) - \frac{(y-3)^2}{2}\right] \\
&= \exp\left[-\frac{1}{2}\left((y-3)^2 - 2\sqrt{2}\,x(y-3)\right)\right] \\
&= \exp\left[-\frac{1}{2}\left((y-3-\sqrt{2}x)^2 - 2x^2\right)\right] \quad \textit{(by completing the square)} \\
&\propto \exp\left[-\frac{1}{2}(y-3-\sqrt{2}x)^2\right]
\end{aligned}
$$

This last expression we recognize as the kernel of a normal density, so we put in the known normalizing constant and we are done. The answer is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(y-3-\sqrt{2}x)^2\right]$$

We can re-state this as $Y|X \sim N(\mu = 3 + \sqrt{2}x\,,\ \sigma^2 = 1)$.

Yet Another Solution:
If the student remembers the bivariate normal density (and realizes that $f(x, y)$ is a bivariate normal density with $\mu_X = 0, \sigma_X^2 = 1/2, \mu_Y = 3, \sigma_Y^2 = 2, \rho = 1/\sqrt{2}$) and also remembers that the bivariate normal has the conditional distribution

$$Y|X \sim N(\mu_Y + \rho(\sigma_Y/\sigma_X)(X - \mu_X), \sigma_Y^2(1 - \rho^2)) \,,$$

then they can just plug the parameter values into this formula to obtain

$$Y|X \sim N\left(3 + (1/\sqrt{2})(\sqrt{2}/\sqrt{1/2})(X - 0), 2(1 - (1/2))\right)$$

so that $Y|X \sim N\left(3 + \sqrt{2}X \,,\, 1\right)$. *This approach, if explained clearly, should receive full credit.*

Problem 4. A random point (X, Y) is distributed uniformly on the region

$$A = \{(x, y) : x > 0, y > 0, x^2 + y^2 < 1\},$$

consisting of the points in the first quadrant which are inside the unit circle.

This problem combines bits of exercises 4.1 and 4.4.

(a) (6%) Let $F(x, y)$ be the joint cdf (cumulative distribution function) of (X, Y). Evaluate $F(c, d)$ where $(c, d) \in A$ is an arbitrary point inside A.

The answer is the ratio of areas $\dfrac{cd}{\mathit{Area}(A)} = \dfrac{cd}{\pi/4}$.

(b) (6%) Find $P(X + Y > 1)$.

The answer is the ratio of areas $\dfrac{(\pi/4) - (1/2)}{\pi/4} = 1 - (2/\pi)$.

Problem 5. Let X and Y be independent random variables with densities

$$f_X(x) = \frac{1}{2}\, e^{-|x|} \quad \text{for} \quad -\infty < x < \infty \quad \text{and}$$

$$f_Y(y) = \frac{1}{\pi(1+y^2)} \quad \text{for} \quad -\infty < y < \infty\,.$$

Define a new random variable Z by

$$Z = \begin{cases} Y & \text{if } XY > 0 \\ -Y & \text{if } XY < 0\,. \end{cases}$$

(a) (10%) What is the density of Z? (Give a precise formula and prove your answer.)

This problem is similar to 4.47(a).

Z has the same density as Y which is $f_Z(z) = \dfrac{1}{\pi(1+z^2)}$.

This problem is the same as 4.47(a) except that the Normal distributions of X and Y have been replaced by other distributions which are symmetric about zero (which is the only property needed to make the arguments work) and the roles of X and Y have been interchanged. The posted solution for exercise 4.47(a) still works so long as X and Y are interchanged everywhere.

(b) (5%) Are Z and X independent? (Answer YES or NO and prove your answer.)

NO, Z and X are NOT independent. Z and X will always have the same sign, so they canNOT be independent. This answer is sufficient and should receive full credit.

A somewhat more formal argument is the following: Both Z and X have densities which are symmetric about zero so that $P(Z<0) = P(Z>0) = P(X<0) = P(X>0) = 1/2$. Since Z and X always have the same sign, we have $P(Z<0, X>0) = 0 \neq 1/4 = P(Z<0)P(X>0)$ which violates independence.

Problem 6. Suppose X and Y are iid (independent and identically distributed) Poisson(λ) random variables.

This problem is similar to exercise 4.15.

(a) (10%) What is the value of the conditional probability $P(X = 2 \,|\, X + Y = 6)$? (Give a detailed argument justifying your answer.)

Students should give an argument similar to the solution of Exercise 4.15 where it is shown that if X, Y are independent with $X \sim$ Poisson(θ) and $Y \sim$ Poisson(λ), then

$$X \,|\, X + Y \sim \textit{Binomial}(n = X + Y, p = \theta/(\theta + \lambda)) .$$

Applying this to the case with $\theta = \lambda$ we obtain

$$\begin{aligned} P(X = 2 \,|\, X + Y = 6) &= P(U = 2) \quad \textit{where } U \sim \textit{Binomial}(n = 6, p = 1/2) \\ &= \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \binom{6}{2}\left(\frac{1}{2}\right)^6 . \end{aligned}$$

The solution manual solution for exercise 4.15 is a little skimpy. A slightly more detailed solution is as follows: For $0 \le i \le j$,

$$\begin{aligned} P(X = i \,|\, X + Y = j) &= \frac{P(X = i, X + Y = j)}{P(X + Y = j)} = \frac{P(X = i, Y = j - i)}{P(X + Y = j)} = \frac{P(X = i)P(Y = j - i)}{P(X + Y = j)} \\ &= \frac{\dfrac{\theta^i e^{-\theta}}{i!}\,\dfrac{\lambda^{j-i} e^{-\lambda}}{(j-i)!}}{\dfrac{(\theta + \lambda)^j e^{-\theta-\lambda}}{j!}} = \binom{j}{i}\left(\frac{\theta}{\theta + \lambda}\right)^i\left(\frac{\lambda}{\theta + \lambda}\right)^{j-i} \\ &= P(U = i) \quad \textit{where } U \sim \textit{Binomial}\left(n = j,\ p = \frac{\theta}{\theta + \lambda}\right) \end{aligned}$$

where in the above we have used $X + Y \sim$ Poisson($\theta + \lambda$). (Students might write the argument only for the special case $\theta = \lambda$, which is fine.)

(b) (5%) Now suppose X, Y, Z are iid Poisson(λ) random variables. What is the value of the conditional probability $P(X = 2 \,|\, X + Y + Z = 6)$?

(Here you will receive full credit just for stating the correct answer. **No work is required. NWR**)

This part is another special case of exercise 4.15. If you define $Y^* = Y + Z$, then the problem becomes $P(X = 2 \,|\, X + Y^* = 6)$.

The answer is: $\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$.

Problem 7. (6%) A bivariate transformation $s = x^2 + y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ has the inverse transformation

$$x = \sqrt{s}\cos\theta\,, \quad y = \sqrt{s}\sin\theta\,.$$

Find the Jacobian of this inverse transformation. (Show your work in detail.)

See page 14 of notes10.pdf for a detailed solution. The answer is: the Jacobian equals 1/2.

Problem 8. (4%) For **discrete** random variables X and Y, the quantity $f_{Y|X}(y|x)$ is the same as _________. (Circle the single correct response.)

a) $\sum_x P(Y = y, X = x)$ **b)** $\dfrac{f_{X,Y}(x,y)\,dx\,dy}{f_Y(y)\,dy}$ **c)** $f_{X,Y}(x,y)\,dx\,dy$ **d)** $\dfrac{P(X = x, Y = y)}{P(Y = y)}$

e)$\star$ $P(Y = y \,|\, X = x)$ **f)** $\dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ **g)** $f_Y(y) f_{X|Y}(x|y)$ **h)** $f_X(x) f_{X|Y}(x|y)$

See page 9 of notes9.pdf.