Assignment #1

Note: Homework exercises are not handed in. However, homework is strongly stressed on the exams. Most of the exam problems will be similar to homework exercises.

Reading: All of Chapter 1 and Section 2.1.

Exercises:

A1, A2, A3 (do these first after reading "Review of Formulas ...")

1.1, 1.2(a,b,d), 1.4, 1.5, 1.6, 1.10, 1.13, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.32, 1.33, 1.34, 1.36, 1.37, 1.38, 1.39, 1.41, 1.44, 1.46, 1.47, 1.51, 1.52, 1.53, 1.54, 1.55

B1 – B8 (do these as the corresponding topics are covered in lecture)

2.1 – 2.9

Some Comments:

1.2, 1.10 Do 1.2(a,b,d) and 1.10 from first principles as in the proof of Theorem 1.1.4 on pages 3 and 4.

1.3, 1.14 You should also read exercises 1.3 and 1.14 and know the results in these exercises.

1.20 Assume the 12 calls are distinct (say, they are from 12 different friends) so that there are 7^{12} equally likely possibilities. Another way to think of this problem is to suppose you have 12 friends and each calls you on a day selected at random. This allows you to use the principle of inclusion-exclusion to solve the problem. This solution is easier than the one in the solution manual.

1.32 The statement of 1.32 is murky and I believe the solution in the manual is wrong. What they are asking for is the following: The N candidates are interviewed in a random order. Suppose the i-th candidate you interview is the best one you have seen so far. What is the (conditional) probability this candidate is the best overall? Assume there are no ties among the candidates, that is, there are no candidates that are rated the same. The correct answer is i/N.

Review of Formulas from Introduction to Probability

- X denotes a random variable (rv).
  x denotes a particular value.

- For a discrete rv, the distribution is described using a probability mass function (pmf) $f_X(x)$ defined by
$$f_X(x) = P(X = x)$$

- For a continuous rv, the distribution is described using a probability density function (pdf) $f_X(x)$ which satisfies
$$P(a \le X \le b) = \int_a^b f_X(x)\, dx$$

- Computing Probabilities
$$\text{discrete rv} \quad P(X \in A) = \sum_{x \in A} f_X(x)$$
$$\text{continuous rv} \quad P(X \in A) = \int_A f_X(x)\, dx$$

- Computing Expected Values
$$\text{discrete rv} \quad EX = \sum_{\text{all } x} x f_X(x)$$
$$\text{continuous rv} \quad EX = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- Computing Expected Values of Functions of X
$$\text{discrete rv} \quad Eg(X) = \sum_{\text{all } x} g(x) f_X(x)$$
$$\text{continuous rv} \quad Eg(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

  so that for example
$$EX^2 = \sum_{\text{all } x} x^2 f_X(x) \quad \text{or} \quad \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

- Properties of Expectation.

  X, Y, Z are rv's with finite expected values. a, b, c are constants (not random).

$$E(aX) = aE(X)$$

$$E(X + b) = E(X) + b$$

$$E(X + Y) = E(X) + E(Y)\,,$$

$$E(X + Y + Z) = E(X) + E(Y) + E(Z)\,, \text{ etc.}$$

  Using these rules in combination gives things like

$$E(aX + bY + c) = aE(X) + bE(Y) + c\,, \text{ etc.}$$

- Variance

$$\mathrm{Var}\, X = E(X - EX)^2 = E(X^2) - (EX)^2$$

- Properties of Variance

  All random variables appearing below are assumed to have a finite variance.

$$\mathrm{Var}\,(aX) = a^2 \mathrm{Var}\,(X)$$

$$\mathrm{Var}\,(X + b) = \mathrm{Var}\,(X)$$

  The following results require that X, Y, Z be **mutually independent** rv's. (They are false in general.)

$$\mathrm{Var}\,(X + Y) = \mathrm{Var}\,(X) + \mathrm{Var}\,(Y)$$

$$\mathrm{Var}\,(X + Y + Z) = \mathrm{Var}\,(X) + \mathrm{Var}\,(Y) + \mathrm{Var}\,(Z)\,, \text{ etc.}$$

  Using these rules in combination gives things like

$$\mathrm{Var}\,(aX + bY + c) = a^2 \mathrm{Var}\,(X) + b^2 \mathrm{Var}\,(Y)$$

Exercises A

These problems are intended to review the basic facts and formulas from introductory probability given in the previous pages.

Problem 1. Let $X \sim \text{Poisson}(1)$. Calculate the following.

(a) $P(X \geq 2)$

(b) $E\left(\frac{1}{X+1}\right)$

Problem 2. Suppose X has pdf defined by $f(x) = cx$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

(a) What is the value of the constant c?

(b) Calculate $P\{1/2 < X < 3/4\}$.

(c) Calculate $E(\log X)$.

(d) Calculate $\text{Var}(X^2)$.

Problem 3. Suppose the random variables $Y_1, Y_2, Y_3, \ldots$ are independent with $P\{Y_i = 0\} = P\{Y_i = 1\} = 1/2$ for all i. Define

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{2^i} = \frac{Y_1}{2} + \frac{Y_2}{4} + \frac{Y_3}{8} + \cdots.$$

Calculate the following.

(a) $E(X)$

(b) $\text{Var}(X)$

Exercises B

A monkey types 6 letters at random. (Each keystroke is independent of the others with all 26 possibilities equally likely.)

Problem 1.

(a) What is the probability the monkey types AHA? (That is, the letters AHA occur as three consecutive letters somewhere in the 6 typed letters.)

(b) What is the probability the monkey types XXXX?

(c) What is the probability the monkey types ART or ALL?

Problem 2.

(a) What is the probability the six typed letters are distinct (there are no repeated letters) <u>and</u> in alphabetical order?

(b) What is the probability the six typed letters are in alphabetical order? (Here we are allowing repetitions so that something like BBBJXX is legal.)

Problem 3. Suppose n people play Russian roulette. Each person has a gun which fires with probability π when the trigger is pulled. (Assume the guns are independent of each other and successive shots of the same gun are independent.) A round of play consists of every one who is still alive raising the guns to their temples and firing simultaneously. Play continues until everyone is dead.

(a) What is the probability that one or more people are still alive after k rounds of play?

(b) The last person (or persons) to die receives a prize (flowers on the grave). What is the probability this prize goes to only one person?
(Note: The answer may not have a simple form, so do not worry if your answer is messy or contains summations you do not know how to do.)

Problem 4. Show that

$$P(A \cap B^c \cap C^c) = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C).$$

Problem 5. What is the probability that a poker hand contains nothing, not even a pair? (You may find the result of the previous problem useful here, depending on how you approach the problem.)

Problem 5.1. Prove the principle of inclusion-exclusion for the case of $k = 4$ events. You may use the result for $k = 2$ and $k = 3$ events in your proof.

Problem 6. Consider the "Gambler's Ruin" problem with a fair coin as discussed in class. Suppose your initial fortune is 10 dollars and your goal is to reach 100 dollars.

(a) What is the probability of achieving the goal?

(b) Given that you reached the goal, what is the probability you lost the first 3 tosses?

(c) Given that you reached the goal, what is the probability your holdings never dropped below 10 dollars? (That is, you were never at any time a net loser.)

Problem 7. Consider the "Gambler's Ruin" problem with a biased coin having probability p of Heads (win a dollar) and $1 - p$ of Tails (lose a dollar). Let $\psi(z)$ denote the probability of reaching a given goal of g dollars starting with an initial fortune of z dollars.

(a) Use the Law of Total Probability to find an equation that $\psi(z)$ must satisfy.

(b) Let $R = (1 - p)/p$. Show that the function

$$\psi(z) = \frac{R^z - 1}{R^g - 1}$$

satisfies the equation found in (a) and also the boundary conditions $\psi(0) = 0$ and $\psi(g) = 1$. (In fact, this function is the unique one satisfying these conditions and must therefore be the answer we want.)

(c) Suppose $p = .48$ and $g = 100$. What is the value of $\psi(10)$? (Compare with the answer in part (a) of the previous problem.)

Problem 8. Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

(a) Describe (in general) what must be done to show that four events A, B, C, D are mutually independent.

(b) Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

(c) Show that A_1, A_2, A_4 *are* mutually independent.