Exercise B5.1
Prove the Principle of Inclusion-Exclusion for 4 sets

One argument exactly parallels the case for $k = 3$ sets done in lecture.

$$\begin{aligned} P(A \cup B \cup C \cup D) &= P((A \cup B \cup C) \cup D) \\ &\quad \text{Apply the case } k = 2. \\ &= P(A \cup B \cup C) + P(D) - P((A \cup B \cup C) \cap D)\,. \quad (\ddagger) \end{aligned}$$

Using the case for $k = 3$ sets gives

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C)\,. \end{aligned}$$

Using the distributive law for sets gives

$$\begin{aligned} &P((A \cup B \cup C) \cap D) \\ &\quad = P((A \cap D) \cup (B \cap D) \cup (C \cap D))\,. \\ &\qquad \text{Apply the case } k = 3. \\ &\quad = P(A \cap D) + P(B \cap D) + P(C \cap D) \\ &\qquad - P((A \cap D) \cap (B \cap D)) - P((A \cap D) \cap (C \cap D)) - P((B \cap D) \cap (C \cap D)) \\ &\qquad + P((A \cap D) \cap (B \cap D) \cap (C \cap D)) \\ &\qquad \text{Apply the associative and commutative laws for } \cap \text{ to several terms above.} \\ &\quad = P(A \cap D) + P(B \cap D) + P(C \cap D) \\ &\qquad - P(A \cap B \cap D) - P(A \cap C \cap D) - P(B \cap C \cap D) \\ &\qquad + P(A \cap B \cap C \cap D)\,. \end{aligned}$$

Plugging these facts back into ($\ddagger$) gives the final result

$$\begin{aligned} &P(A \cup B \cup C \cup D) \\ &\quad = P(A) + P(B) + P(C) + P(D) \\ &\qquad - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) \\ &\qquad + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) \\ &\qquad - P(A \cap B \cap C \cap D)\,. \end{aligned}$$