B7 Part b
Morgan Weiss

Problem B7 part b - Suppose

$$\psi(z) = p\psi(z+1) + (1-p)\psi(z-1) \quad \text{for all integers } z \text{ satisfying } 0 < z < g$$

where $0 < p \leq 1$. Suppose also that $\psi(0) = 0$ and $\psi(g) = 1$. Then:
1. if $p = 1/2$, we have

$$\psi(z) = \frac{z}{g}$$

2. if $p \neq 1/2$, we have

$$\psi(z) = \frac{R^z - 1}{R^g - 1} \qquad \text{where } R = \frac{(1-p)}{p}$$

Proof. We have that

$$p\psi(z) + (1-p)\psi(z) = \psi(z) = p\psi(z+1) + (1-p)\psi(z-1)$$

or

$$(1-p)\left(\psi(z) - \psi(z-1)\right) = p\left(\psi(z+1) - \psi(z)\right)$$

dividing both sides by p we get

$$\frac{(1-p)}{p}\left(\psi(z) - \psi(z-1)\right) = \psi(z+1) - \psi(z)$$

that is

$$R\left(\psi(z) - \psi(z-1)\right) = \psi(z+1) - \psi(z)$$

So we have proved that, for $0 < z < g$,

$$\psi(z+1) - \psi(z) = R\left(\psi(z) - \psi(z-1)\right)$$

So we have by a simple induction that

$$\psi(z+1) - \psi(z) = R\left(\psi(z) - \psi(z-1)\right) = R^2\left(\psi(z-1) - \psi(z-2)\right) = \ldots = R^z\left(\psi(1) - \psi(0)\right)$$

Since $\psi(0) = 0$, we have, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z+1) - \psi(z) = R^z\psi(1)$$

Now note that, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = \psi(1) + \sum_{i=1}^{z-1}(\psi(i+1) - \psi(i)) = \psi(1) + \sum_{i=1}^{z-1} R^i\psi(1) = \sum_{i=0}^{z-1} R^i\psi(1)$$

So we have proved that, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = \left(\sum_{i=0}^{z-1} R^i\right)\psi(1) \qquad (1)$$

Now, we have two cases:
Case 1: $p = 1/2$. Then $R = 1$ and we have $\sum_{i=0}^{z-1} R^i = z$. So we have, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = z\psi(1)$$

In particular, for $z = g$, since $\psi(g) = 1$, we have

$$1 = \psi(g) = g\psi(1)$$

Since $\psi(0) = 0$ and $\psi(g) = 1$, we have that $g \neq 0$ and

$$\psi(1) = \frac{1}{g}$$

So we have, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = \frac{z}{g}$$

For $z = 0$ we trivially have

$$\psi(0) = 0 = \frac{0}{g}$$

So we have that, for any $z \in \{0, 1, \ldots, g\}$,

$$\psi(z) = \frac{z}{g}$$

Case 2. $p \neq 1/2$. Then $R \neq 1$ and we have

$$\sum_{i=0}^{z-1} R^i = \frac{R^z - 1}{R - 1}$$

So, from (1), we have, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = \frac{R^z - 1}{R - 1}\psi(1)$$

In particular, for $z = g$, since $\psi(g) = 1$, we have

$$1 = \psi(g) = \frac{R^g - 1}{R - 1}\psi(1)$$

So

$$\psi(1) = \frac{R - 1}{R^g - 1}$$

So we have, for any $z \in \{1, 2, \ldots, g\}$,

$$\psi(z) = \frac{R^z - 1}{R^g - 1}$$

For $z = 0$ we trivially have

$$\psi(0) = 0 = \frac{R^0 - 1}{R^g - 1}$$

So we have that, for any $z \in \{0, 1, \ldots, g\}$,

$$\psi(z) = \frac{R^z - 1}{R^g - 1}$$

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