Problem 2.11(b)

There are a number of typos in the solution manual for this problem. They are corrected below.

$Y = |X|$ where $-\infty < X < \infty$. Therefore $0 < Y < \infty$. For $y > 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) \\ &= P(X \leq y) - P(X \leq -y) = F_X(y) - F_X(-y)\,, \end{aligned}$$

and $F_Y(y) = 0$ for $y \leq 0$. Therefore,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X(y) + f_X(-y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(-y)^2}{2}} = \sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}}$$

for $y > 0$ (and $f_Y(y) = 0$ for $y < 0$). Thus,

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy = \sqrt{\frac{2}{\pi}}\int_0^{\infty} ye^{-\frac{y^2}{2}}\,dy = \sqrt{\frac{2}{\pi}}\left(-e^{-y^2/2}\right)\Big|_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

and

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \sqrt{\frac{2}{\pi}}\int_0^{\infty} y^2 e^{-y^2/2}dy$$

making the substitution $u = y^2/2$, $y = \sqrt{2u}$, $dy = \dfrac{du}{\sqrt{2u}}$

$$\begin{aligned} &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}(2u)e^{-u}\frac{du}{\sqrt{2u}} \\ &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} u^{\frac{1}{2}}e^{-u}du = \frac{2}{\sqrt{\pi}}\int_0^{\infty} u^{\frac{3}{2}-1}e^{-u}du = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{2}\sqrt{\pi}\right) = 1\,. \end{aligned}$$

(Instead of using the substitution, one can integrate by parts as in the solution manual.) Therefore, we can compute the variance as:

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = 1 - \left(\sqrt{\frac{2}{\pi}}\right)^2 = 1 - \frac{2}{\pi}\,.$$