Exercise 2.27(c)

Problem: Show that if $f(x)$ is both symmetric (see Exercise 2.26) and unimodal, then the point of symmetry is a mode.

Solution: Let f be symmetric about the point s and unimodal with mode m. If $s = m$, then s is also a mode and we are done. So we need only consider the situation where $s \neq m$.

Some useful notation: For any real number x, let x' denote the reflection of x about s. The points x and x' are on opposite sides of s and are equally distant from x; if $x = s + u$ for some u, then $x' = s - u$. Since f is symmetric about s, we know that $f(x) = f(x')$ for all x. It is obvious that $x'' = x$.

Lemma: m' is also a mode of f.

Comment: It is easy to show that f is constant (flat) between m and m', but we will not prove this since we do not need it below.

Proof of Lemma: By the definition of unimodal in the problem statement, to show that m' is a mode we must show that if $m' \geq x \geq y$ or $m' \leq x \leq y$, then $f(m') \geq f(x) \geq f(y)$. Suppose $m' \geq x \geq y$. By reflecting all these points, it is clear that $m \leq x' \leq y'$. Since m is a mode, this implies that $f(m) \geq f(x') \geq f(y')$ which implies $f(m') \geq f(x) \geq f(y)$ as desired. (Here we have used $f(x) = f(x')$ and $x'' = x$ for all x.) The case where $m' \leq x \leq y$ is handled in the same way.

Proof of Main Result: Now we are ready to prove that s is a mode. We must show that if $s \geq x \geq y$ or $s \leq x \leq y$, then $f(s) \geq f(x) \geq f(y)$. Suppose first that $s \geq x \geq y$. Since $s \neq m$, we know that either $m > s$ or $m < s$. Suppose that $m > s$, then we know $m \geq s \geq x \geq y$. Since m is a mode, this implies $f(m) \geq f(s) \geq f(x) \geq f(y)$ which implies $f(s) \geq f(x) \geq f(y)$ as desired. (See note below.) Alternatively, if $m < s$ then $m' > s$. But m' is also a mode, so we can use exactly the same argument but with m' instead of m to again show $f(s) \geq f(x) \geq f(y)$.

The case where $s \leq x \leq y$ is handled in the same way (and could perhaps be omitted). If $m < s$, then $m \leq s \leq x \leq y$ so that $f(m) \geq f(s) \geq f(x) \geq f(y)$ and thus $f(s) \geq f(x) \geq f(y)$ as desired. If $m > s$, then $m' < s$ and use the same argument with m' instead of m.

Note: Strictly speaking, the definition of mode does not apply to inequalities involving four quantities such as $m \geq s \geq x \geq y$, but only to inequalities involving three quantities like $m \geq s \geq x$. But we can always argue in pieces and chain the results together: $m \geq s \geq x \geq y$ implies both $m \geq s \geq x$ and $m \geq x \geq y$ which implies $f(m) \geq f(s) \geq f(x)$ and $f(m) \geq f(x) \geq f(y)$ which implies $f(s) \geq f(x) \geq f(y)$ as desired.