Prove:

$$\sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k} = (n-x)\binom{n}{x}\int_0^{1-p} t^{n-x-1}(1-t)^x dt$$

Let

$$I_i = (n-x+i)\binom{n}{x-i}\int_0^{1-p} t^{n-x+i-1}(1-t)^{x-i}dt$$

The RHS of the original expression is I_0, which we repeatedly integrate by parts to obtain the terms in the LHS sum:

$$\begin{aligned}
I_0 &= (n-x)\binom{n}{x}\left[\frac{1}{n-x}t^{n-x}(1-t)^x\Big|_0^{1-p} + \int_0^{1-p}\frac{x}{n-x}t^{n-x}(1-t)^{x-1}dx\right] \\
&= \binom{n}{x}(1-p)^{n-x}p^x + x\binom{n}{x}\int_0^{1-p} t^{n-x}(1-t)^{x-1}dt \\
&= \binom{n}{x}p^x(1-p)^{n-x} + (n-x+1)\binom{n}{x-1}\int_0^{1-p} t^{n-x}(1-t)^{x-1}dt
\end{aligned}$$

This is the $k = x$ term of the sum plus I_1. For $i = 1, 2, ..., x-1$, integration by parts produces:

$$\begin{aligned}
I_i &= (n-x+i)\binom{n}{x-i}\left[\frac{1}{n-x+i}t^{n-x+i}(1-t)^{x-i}\Big|_0^{1-p} + \int_0^{1-p}\frac{x-i}{n-x+i}t^{n-x+i}(1-t)^{x-i-1}dt\right] \\
&= \binom{n}{x-i}(1-p)^{n-x+i}p^{x-i} + (x-i)\binom{n}{x-i}\int_0^{1-p} t^{n-x+i}(1-t)^{x-i-1}dt \\
&= \binom{n}{x-i}p^{x-i}(1-p)^{n-(x-i)} + (n-x+(i+1))\binom{n}{x-(i+1)}\int_0^{1-p} t^{n-x+(i+1)-1}(1-t)^{x-(i+1)}dt
\end{aligned}$$

For each i, this gives I_{i+1} and the $k = x - i$ term of the sum. Putting it all together after $i = x - 1$, we have

$$\begin{aligned}
(n-x)\binom{n}{x}\int_0^{1-p} t^{n-x-1}(1-t)^x dt &= \left(\sum_{k=1}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right) + I_x \\
&= \left(\sum_{k=1}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right) + n\binom{n}{0}\int_0^{1-p} t^{n-1}(1-t)^0 dt \\
&= \left(\sum_{k=1}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right) + n\left(\frac{1}{n}t^n\right)\Big|_0^{1-p} \\
&= \left(\sum_{k=1}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right) + (1-p)^n \\
&= \sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}
\end{aligned}$$

which is the desired result.