2.9 From the probability integral transformation, Theorem 2.1.10, we know that if $u(x) = F_x(x)$, then $F_x(X) \sim$ uniform$(0,1)$. Therefore, for the given pdf, calculate

$$u(x) = F_x(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ (x-1)^2/4 & \text{if } 1 < x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

2.10 a. We prove part b), which is equivalent to part a).

b. Let $A_y = \{x : F_x(x) \leq y\}$. Since F_x is nondecreasing, A_y is a half infinite interval, either open, say $(-\infty, x_y)$, or closed, say $(-\infty, x_y]$. If A_y is closed, then

$$F_Y(y) = P(Y \leq y) = P(F_x(X) \leq y) = P(X \in A_y) = F_x(x_y) \leq y.$$

The last inequality is true because $x_y \in A_y$, and $F_x(x) \leq y$ for every $x \in A_y$. If A_y is open, then

$$F_Y(y) = P(Y \leq y) = P(F_x(X) \leq y) = P(X \in A_y),$$

as before. But now we have

$$P(X \in A_y) = P(X \in (-\infty, x_y)) = \lim_{x \uparrow y} P(X \in (-\infty, x]),$$

Use the Axiom of Continuity, Exercise 1.12, and this equals $\lim_{x\uparrow y} F_X(x) \leq y$. The last inequality is true since $F_x(x) \leq y$ for every $x \in A_y$, that is, for every $x < x_y$. Thus, $F_Y(y) \leq y$ for every y. To get strict inequality for some y, let y be a value that is "jumped over" by F_x. That is, let y be such that, for some x_y,

$$\lim_{x \uparrow y} F_X(x) < y < F_X(x_y).$$

For such a y, $A_y = (-\infty, x_y)$, and $F_Y(y) = \lim_{x\uparrow y} F_X(x) < y$.

2.11 a. Using integration by parts with $u = x$ and $dv = xe^{\frac{-x^2}{2}} dx$ then

$$\mathrm{E}X^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{2\pi} e^{\frac{-x^2}{2}} dx = \frac{1}{2\pi}\left[-xe^{\frac{-x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \right] = \frac{1}{2\pi}(2\pi) = 1.$$

Using example 2.1.7 let $Y = X^2$. Then

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[\frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} \right] = \frac{1}{\sqrt{2\pi y}} e^{\frac{-y}{2}}.$$

Therefore,

$$\mathrm{E}Y = \int_0^{\infty} \frac{y}{\sqrt{2\pi y}} e^{\frac{-y}{2}} dy = \frac{1}{\sqrt{2\pi}} \left[-2y^{\frac{1}{2}} e^{\frac{-y}{2}} \Big|_0^{\infty} + \int_0^{\infty} y^{\frac{-1}{2}} e^{\frac{-y}{2}} dy \right] = \frac{1}{\sqrt{2\pi}}(\sqrt{2\pi}) = 1.$$

This was obtained using integration by parts with $u = 2y^{\frac{1}{2}}$ and $dv = \frac{1}{2} e^{\frac{-y}{2}}$ and the fact the $f_Y(y)$ integrates to 1.

b. $Y = |X|$ where $-\infty < x < \infty$. Therefore $0 < y < \infty$. Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) \\ &= P(x \leq y) - P(X \leq -y) = F_X(y) - F_X(-y). \end{aligned}$$

Therefore,

$$F_Y(y) = \frac{d}{dy}F_Y(y) = f_X(y) + f_X(-y) = \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}} = \sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}.$$

Thus,

$$\mathrm{E}Y = \int_0^\infty y\sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}dy = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u}du = \sqrt{\frac{2}{\pi}}\left[-e^{-u}\Big|_0^\infty\right] = \sqrt{\frac{2}{\pi}},$$

where $u = \frac{y^2}{2}$.

$$\mathrm{E}Y^2 = \int_0^\infty y^2\sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}dy = \sqrt{\frac{2}{\pi}}\left[-ye^{\frac{-y}{2}}\Big|_0^\infty + \int_0^\infty e^{\frac{-y}{2}}dy\right] = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}} = 1.$$

This was done using integration by part with $u = y$ and $dv = ye^{\frac{-y}{2}}dy$. Then $\mathrm{Var}(Y) = 1 - \frac{2}{\pi}$.

2.12 We have $\tan x = y/d$, therefore $\tan^{-1}(y/d) = x$ and $\frac{d}{dy}\tan^{-1}(y/d) = \frac{1}{1+(y/d)^2}\frac{1}{d}dy = dx$. Thus,

$$f_Y(y) = \frac{2}{\pi d}\frac{1}{1+(y/d)^2}, \quad 0 < y < \infty.$$

This is the Cauchy distribution restricted to $(0,\infty)$, and the mean is infinite.

2.13 $P(X = k) = (1-p)^k p + p^k(1-p)$, $k = 1, 2, \ldots$. Therefore,

$$\begin{aligned}
\mathrm{E}X &= \sum_{k=1}^\infty k[(1-p)^k p + p^k(1-p)] = (1-p)p\left[\sum_{k=1}^\infty k(1-p)^{k-1} + \sum_{k=1}^\infty kp^{k-1}\right] \\
&= (1-p)p\left[\frac{1}{p^2} + \frac{1}{(1-p)^2}\right] = \frac{1-2p+2p^2}{p(1-p)}.
\end{aligned}$$

2.14

$$\begin{aligned}
\int_0^\infty (1-F_X(x))dx &= \int_0^\infty P(X > x)dx \\
&= \int_0^\infty \int_x^\infty f_X(y)dydx \\
&= \int_0^\infty \int_0^y dx f_X(y)dy \\
&= \int_0^\infty y f_X(y)dy = \mathrm{E}X,
\end{aligned}$$

where the last equality follows from changing the order of integration.

2.15 Assume without loss of generality that $X \le Y$. Then $X \vee Y = Y$ and $X \wedge Y = X$. Thus $X + Y = (X \wedge Y) + (X \vee Y)$. Taking expectations

$$\mathrm{E}[X+Y] = \mathrm{E}[(X \wedge Y) + (X \vee Y)] = \mathrm{E}(X \wedge Y) + \mathrm{E}(X \vee Y).$$

Therefore $\mathrm{E}(X \vee Y) = \mathrm{E}X + \mathrm{E}Y - \mathrm{E}(X \wedge Y)$.

2.16 From Exercise 2.14,

$$\mathrm{E}T = \int_0^\infty \left[ae^{-\lambda t} + (1-a)e^{-\mu t}\right]dt = \frac{-ae^{-\lambda t}}{\lambda} - \frac{(1-a)e^{-\mu t}}{\mu}\Big|_0^\infty = \frac{a}{\lambda} + \frac{1-a}{\mu}.$$

2.17 a. $\int_0^m 3x^2dx = m^3 \stackrel{set}{=} \frac{1}{2} \quad \Rightarrow \quad m = \left(\frac{1}{2}\right)^{1/3} = .794.$

b. The function is symmetric about zero, therefore $m = 0$ as long as the integral is finite.

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1.$$

This is the Cauchy pdf.

2.18 $\mathrm{E}|X-a| = \int_{-\infty}^{\infty}|x-a|f(x)dx = \int_{-\infty}^{a}-(x-a)f(x)dx + \int_a^{\infty}(x-a)f(x)dx$. Then,

$$\frac{d}{da}\mathrm{E}|X-a| = \int_{-\infty}^{a} f(x)dx - \int_a^{\infty} f(x)dx \stackrel{\text{set}}{=} 0.$$

The solution to this equation is $a =$ median. This is a minimum since $d^2/da^2\mathrm{E}|X-a| = 2f(a) > 0$.

2.19

$$\begin{aligned}
\frac{d}{da}\mathrm{E}(X-a)^2 &= \frac{d}{da}\int_{-\infty}^{\infty}(x-a)^2 f_X(x)dx &&= \int_{-\infty}^{\infty}\frac{d}{da}(x-a)^2 f_X(x)dx \\
&= \int_{-\infty}^{\infty}-2(x-a)f_X(x)dx &&= -2\left[\int_{-\infty}^{\infty}xf_X(x)dx - a\int_{-\infty}^{\infty}f_X(x)dx\right] \\
&= -2[\mathrm{E}X-a].
\end{aligned}$$

Therefore if $\frac{d}{da}\mathrm{E}(X-a)^2 = 0$ then $-2[\mathrm{E}X-a] = 0$ which implies that $\mathrm{E}X = a$. If $\mathrm{E}X = a$ then $\frac{d}{da}\mathrm{E}(X-a)^2 = -2[\mathrm{E}X-a] = -2[a-a] = 0$. $\mathrm{E}X = a$ is a minimum since $d^2/da^2\mathrm{E}(X-a)^2 = 2 > 0$. The assumptions that are needed are the ones listed in Theorem 2.4.3.

2.20 From Example 1.5.4, if $X =$ number of children until the first daughter, then

$$P(X=k) = (1-p)^{k-1}p,$$

where p = probability of a daughter. Thus X is a geometric random variable, and

$$\begin{aligned}
\mathrm{E}X &= \sum_{k=1}^{\infty}k(1-p)^{k-1}p &&= p-\sum_{k=1}^{\infty}\frac{d}{dp}(1-p)^k &&= -p\frac{d}{dp}\left[\sum_{k=0}^{\infty}(1-p)^k-1\right] \\
&= -p\frac{d}{dp}\left[\frac{1}{p}-1\right] &&= \frac{1}{p}.
\end{aligned}$$

Therefore, if p $= \frac{1}{2}$,the expected number of children is two.

2.21 Since $g(x)$ is monotone

$$\mathrm{E}g(X) = \int_{-\infty}^{\infty} g(x)f_X(x)dx = \int_{-\infty}^{\infty} yf_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)dy = \int_{-\infty}^{\infty} yf_Y(y)dy = \mathrm{E}Y,$$

where the second equality follows from the change of variable $y = g(x)$, $x = g^{-1}(y)$ and $dx = \frac{d}{dy}g^{-1}(y)dy$.

2.22 a. Using integration by parts with $u = x$ and $dv = xe^{-x^2/\beta^2}$ we obtain that

$$\int_0^{\infty} x^2e^{-x^2/\beta^2}dx^2 = \frac{\beta^2}{2}\int_0^{\infty} e^{-x^2/\beta^2}dx.$$

The integral can be evaluated using the argument on pages 104-105 (see 3.3.14) or by transforming to a gamma kernel (use $y = -\lambda^2/\beta^2$). Therefore, $\int_0^{\infty} e^{-x^2/\beta^2}dx = \sqrt{\pi}\beta/2$ and hence the function integrates to 1.

b. $\mathrm{E}X = 2\beta/\sqrt{\pi}$ $\qquad$ $\mathrm{E}X^2 = 3\beta^2/2$ $\qquad$ $\mathrm{Var}X = \beta^2\left[\frac{3}{2} - \frac{4}{\pi}\right]$.

2.23 a. Use Theorem 2.1.8 with $A_0 = \{0\}$, $A_1 = (-1, 0)$ and $A_2 = (0, 1)$. Then $g_1(x) = x^2$ on A_1 and $g_2(x) = x^2$ on A_2. Then

$$f_Y(y) = \frac{1}{2}y^{-1/2}, \quad 0 < y < 1.$$

b. $\mathrm{E}Y = \int_0^1 y f_Y(y)dy = \frac{1}{3}$ $\qquad$ $\mathrm{E}Y^2 = \int_0^1 y^2 f_Y(y)dy = \frac{1}{5}$ $\qquad$ $\mathrm{Var}Y = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$.

2.24 a. $\mathrm{E}X = \int_0^1 x a x^{a-1}dx = \int_0^1 a x^a dx = \left.\frac{ax^{a+1}}{a+1}\right|_0^1 = \frac{a}{a+1}$.

$\mathrm{E}X^2 = \int_0^1 x^2 a x^{a-1}dx = \int_0^1 a x^{a+1} dx = \left.\frac{ax^{a+2}}{a+2}\right|_0^1 = \frac{a}{a+2}$.

$\mathrm{Var}X = \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2 = \frac{a}{(a+2)(a+1)^2}$.

b. $\mathrm{E}X = \sum_{x=1}^n \frac{x}{n} = \frac{1}{n}\sum_{x=1}^n x = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$.

$\mathrm{E}X^2 = \sum_{i=1}^n \frac{x^2}{n} = \frac{1}{n}\sum_{i=1}^n x^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$.

$\mathrm{Var}X = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2+1}{12}$.

c. $\mathrm{E}X = \int_0^2 x\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2 (x^3 - 2x^2 + x)dx = 1$.

$\mathrm{E}X^2 = \int_0^2 x^2\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2 (x^4 - 2x^3 + x^2)dx = \frac{8}{5}$.

$\mathrm{Var}X = \frac{8}{5} - 1^2 = \frac{3}{5}$.

2.25 a. $Y = -X$ and $g^{-1}(y) = -y$. Thus $f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| = f_X(-y)|-1| = f_X(y)$ for every y.

b. To show that $M_X(t)$ is symmetric about 0 we must show that $M_X(0+\epsilon) = M_X(0-\epsilon)$ for all $\epsilon > 0$.

$$\begin{aligned} M_X(0+\epsilon) &= \int_{-\infty}^{\infty} e^{(0+\epsilon)x}f_X(x)dx = \int_{-\infty}^{0} e^{\epsilon x}f_X(x)dx + \int_{0}^{\infty} e^{\epsilon x}f_X(x)dx \\ &= \int_{0}^{\infty} e^{\epsilon(-x)}f_X(-x)dx + \int_{-\infty}^{0} e^{\epsilon(-x)}f_X(-x)dx = \int_{-\infty}^{\infty} e^{-\epsilon x}f_X(x)dx \\ &= \int_{-\infty}^{\infty} e^{(0-\epsilon)x}f_X(x)dx = M_X(0-\epsilon). \end{aligned}$$

2.26 a. There are many examples; here are three. The standard normal pdf (Example 2.1.9) is symmetric about $a = 0$ because $(0-\epsilon)^2 = (0+\epsilon)^2$. The Cauchy pdf (Example 2.2.4) is symmetric about $a = 0$ because $(0-\epsilon)^2 = (0+\epsilon)^2$. The uniform$(0,1)$ pdf (Example 2.1.4) is symmetric about $a = 1/2$ because

$$f((1/2)+\epsilon) = f((1/2)-\epsilon) = \begin{cases} 1 & \text{if } 0 < \epsilon < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le \epsilon < \infty \end{cases}.$$

b.

$$\begin{aligned} \int_a^{\infty} f(x)dx &= \int_0^{\infty} f(a+\epsilon)d\epsilon && \text{(change variable, } \epsilon = x - a\text{)} \\ &= \int_0^{\infty} f(a-\epsilon)d\epsilon && (f(a+\epsilon) = f(a-\epsilon) \text{ for all } \epsilon > 0) \\ &= \int_{-\infty}^{a} f(x)dx. && \text{(change variable, } x = a - \epsilon\text{)} \end{aligned}$$

Since

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1,$$

it must be that

$$\int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = 1/2.$$

Therefore, a is a median.

c.

$$\begin{aligned} \mathrm{E}X - a &= \mathrm{E}(X-a) = \int_{-\infty}^{\infty} (x-a)f(x)dx \\ &= \int_{-\infty}^{a} (x-a)f(x)dx + \int_{a}^{\infty} (x-a)f(x)dx \\ &= \int_{0}^{\infty} (-\epsilon)f(a-\epsilon)d\epsilon + \int_{0}^{\infty} \epsilon f(a+\epsilon)d\epsilon \end{aligned}$$

With a change of variable, $\epsilon = a - x$ in the first integral, and $\epsilon = x - a$ in the second integral we obtain that

$$\begin{aligned} \mathrm{E}X - a &= \mathrm{E}(X-a) \\ &= -\int_{0}^{\infty} \epsilon f(a-\epsilon)d\epsilon + \int_{0}^{\infty} \epsilon f(a-\epsilon)d\epsilon \qquad (f(a+\epsilon) = f(a-\epsilon) \text{ for all } \epsilon > 0) \\ &= 0. \qquad \text{(two integrals are same)} \end{aligned}$$

Therefore, $\mathrm{E}X = a$.

d. If $a > \epsilon > 0$,

$$f(a-\epsilon) = e^{-(a-\epsilon)} > e^{-(a+\epsilon)} = f(a+\epsilon).$$

Therefore, $f(x)$ is not symmetric about $a > 0$. If $-\epsilon < a \le 0$,

$$f(a-\epsilon) = 0 < e^{-(a+\epsilon)} = f(a+\epsilon).$$

Therefore, $f(x)$ is not symmetric about $a \le 0$, either.

e. The median of $X = \log 2 < 1 = \mathrm{E}X$.

2.27 a. The standard normal pdf.

b. The uniform on the interval $(0,1)$.

c. For the case when the mode is unique. Let a be the point of symmetry and b be the mode. Let assume that a is not the mode and without loss of generality that $a = b+\epsilon > b$ for $\epsilon > 0$. Since b is the mode then $f(b) > f(b+\epsilon) \ge f(b+2\epsilon)$ which implies that $f(a-\epsilon) > f(a) \ge f(a+\epsilon)$ which contradict the fact the $f(x)$ is symmetric. Thus a is the mode.
For the case when the mode is not unique, there must exist an interval (x_1, x_2) such that $f(x)$ has the same value in the whole interval, i.e, $f(x)$ is flat in this interval and for all $b \in (x_1, x_2)$, b is a mode. Let assume that $a \notin (x_1, x_2)$, thus a is not a mode. Let also assume without loss of generality that $a = (b+\epsilon) > b$. Since b is a mode and $a = (b+\epsilon) \notin (x_1, x_2)$ then $f(b) > f(b+\epsilon) \ge f(b+2\epsilon)$ which contradict the fact the $f(x)$ is symmetric. Thus $a \in (x_1, x_2)$ and is a mode.

d. $f(x)$ is decreasing for $x \ge 0$, with $f(0) > f(x) > f(y)$ for all $0 < x < y$. Thus $f(x)$ is unimodal and 0 is the mode.

2.28 a.

$$\begin{aligned}
\mu_3 &= \int_{-\infty}^{\infty}(x-a)^3 f(x)dx \;=\; \int_{-\infty}^{a}(x-a)^3 f(x)dx + \int_{a}^{\infty}(x-a)^3 f(x)dx \\
&= \int_{-\infty}^{0} y^3 f(y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \qquad \text{(change variable } y = x-a) \\
&= \int_{0}^{\infty} -y^3 f(-y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \\
&= 0. \qquad\qquad (f(-y+a) = f(y+a))
\end{aligned}$$

b. For $f(x) = e^{-x}$, $\mu_1 = \mu_2 = 1$, therefore $\alpha_3 = \mu_3$.

$$\begin{aligned}
\mu_3 &= \int_0^{\infty}(x-1)^3 e^{-x}dx \;=\; \int_0^{\infty}(x^3 - 3x^2 + 3x - 1)e^{-x}dx \\
&= \Gamma(4) - 3\Gamma(3) + 3\Gamma(2) - \Gamma(1) \;=\; 3! - 3\times 2! + 3\times 1 - 1 \;=\; 2.
\end{aligned}$$

c. Each distribution has $\mu_1 = 0$, therefore we must calculate $\mu_2 = \mathrm{E}X^2$ and $\mu_4 = \mathrm{E}X^4$.

(i) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $\mu_2 = 1$, $\mu_4 = 3$, $\alpha_4 = 3$.
(ii) $f(x) = \frac{1}{2}$, $-1 < x < 1$, $\mu_2 = \frac{1}{3}$, $\mu_4 = \frac{1}{5}$, $\alpha_4 = \frac{9}{5}$.
(iii) $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$, $\mu_2 = 2$, $\mu_4 = 24$, $\alpha_4 = 6$.

As a graph will show, (iii) is most peaked, (i) is next, and (ii) is least peaked.

2.29 a. For the binomial

$$\begin{aligned}
\mathrm{E}X(X-1) &= \sum_{x=2}^{n} x(x-1)\binom{n}{x}p^x(1-p)^{n-x} \\
&= n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x}p^{x-2}(1-p)^{n-x} \\
&= n(n-1)p^2\sum_{y=0}^{n-2}\binom{n-2}{y}p^y(1-p)^{n-2-y} = n(n-1)p^2,
\end{aligned}$$

where we use the identity $x(x-1)\binom{n}{x} = n(n-1)\binom{n-2}{x}$, substitute $y = x-2$ and recognize that the new sum is equal to 1. Similarly, for the Poisson

$$\mathrm{E}X(X-1) = \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2,$$

where we substitute $y = x-2$.

b. $\mathrm{Var}(X) = \mathrm{E}[X(X-1)] + \mathrm{E}X - (\mathrm{E}X)^2$. For the binomial

$$\mathrm{Var}(X) = n(n-1)p^2 + np - (np)^2 = np(1-p).$$

For the Poisson

$$\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

c.

$$\mathrm{E}Y = \sum_{y=0}^{n} y\frac{a}{y+a}\binom{n}{y}\frac{\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}} = \sum_{y=1}^{n} n\frac{a}{(y-1)+(a+1)}\binom{n-1}{y-1}\frac{\binom{a+b-1}{a}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}}$$

$$
\begin{aligned}
&= \sum_{y=1}^{n} n \frac{a}{(y-1)+(a+1)} \binom{n-1}{y-1} \frac{\binom{a+b-1}{a}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}} \\
&= \frac{\frac{na}{a+1}\binom{a+b-1}{a}}{\binom{a+1+b-1}{a+1}} \sum_{y=1}^{n} \frac{a+1}{(y-1)+(a+1)} \binom{n-1}{y-1} \frac{\binom{a+1+b-1}{a+1}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}} \\
&= \frac{na}{a+b} \sum_{j=0}^{n-1} \frac{a+1}{j+(a+1)} \binom{n-1}{j} \frac{\binom{a+1+b-1}{a+1}}{\binom{(n-1)+(a+1)+b-1}{(j+(a+1)}} = \frac{na}{a+b},
\end{aligned}
$$

since the last summation is 1, being the sum over all possible values of a beta-binomial$(n-1, a+1, b)$. $\mathrm{E}[Y(Y-1)] = \frac{n(n-1)a(a+1)}{(a+b)(a+b+1)}$ is calculated similar to EY, but using the identity $y(y-1)\binom{n}{y} = n(n-1)\binom{n-2}{y-2}$ and adding 2 instead of 1 to the parameter a. The sum over all possible values of a beta-binomial$(n-2, a+2, b)$ will appear in the calculation. Therefore

$$\mathrm{Var}(Y) = \mathrm{E}[Y(Y-1)] + \mathrm{E}Y - (\mathrm{E}Y)^2 = \frac{nab(n+a+b)}{(a+b)^2(a+b+1)}.$$

2.30 a. $\mathrm{E}(e^{tX}) = \int_0^c e^{tx}\frac{1}{c}dx = \frac{1}{ct}e^{tx}\Big|_0^c = \frac{1}{ct}e^{tc} - \frac{1}{ct}1 = \frac{1}{ct}(e^{tc}-1)$.

b. $\mathrm{E}(e^{tX}) = \int_0^c \frac{2x}{c^2}e^{tx}dx = \frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1)$. (integration-by-parts)

c.

$$
\begin{aligned}
\mathrm{E}(e^{tx}) &= \int_{-\infty}^{\alpha} \frac{1}{2\beta} e^{(x-\alpha)/\beta} e^{tx} dx + \int_{\alpha}^{\infty} \frac{1}{2\beta} e^{-(x-\alpha)/\beta} e^{tx} dx \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \frac{1}{(\frac{1}{\beta}+t)} e^{x(\frac{1}{\beta}+t)} \Bigg|_{-\infty}^{\alpha} + -\frac{e^{\alpha/\beta}}{2\beta} \frac{1}{(\frac{1}{\beta}-t)} e^{-x(\frac{1}{\beta}-t)} \Bigg|_{\alpha}^{\infty} \\
&= \frac{e^{\alpha t}}{1-\beta^2 t^2}, \quad -1/\beta < t < 1/\beta.
\end{aligned}
$$

d. $\mathrm{E}\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x = p^r \sum_{x=0}^{\infty}\binom{r+x-1}{x}\left((1-p)e^t\right)^x$. Now use the fact that $\sum_{x=0}^{\infty}\binom{r+x-1}{x}\left((1-p)e^t\right)^x\left(1-(1-p)e^t\right)^r = 1$ for $(1-p)e^t < 1$, since this is just the sum of this pmf, to get $\mathrm{E}(e^{tX}) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$.

2.31 Since the mgf is defined as $M_X(t) = \mathrm{E}e^{tX}$, we necessarily have $M_X(0) = \mathrm{E}e^0 = 1$. But $t/(1-t)$ is 0 at $t=0$, therefore it cannot be an mgf.

2.32

$$\frac{d}{dt}S(t)\Bigg|_{t=0} = \frac{d}{dt}\left(\log(M_x(t))\right)\Bigg|_{t=0} = \frac{\frac{d}{dt}M_x(t)}{M_x(t)}\Bigg|_{t=0} = \frac{\mathrm{E}X}{1} = \mathrm{E}X \quad (\text{since } M_X(0) = \mathrm{E}e^0 = 1)$$

$$
\begin{aligned}
\frac{d^2}{dt^2}S(t)\Bigg|_{t=0} &= \frac{d}{dt}\left(\frac{M_x'(t)}{M_x(t)}\right)\Bigg|_{t=0} = \frac{M_x(t)M_x''(t) - [M_x'(t)]^2}{[M_x(t)]^2}\Bigg|_{t=0} \\
&= \frac{1\cdot \mathrm{E}X^2 - (\mathrm{E}X)^2}{1} = \mathrm{Var}X.
\end{aligned}
$$

2.33 a. $M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$.

$\mathrm{E}X = \frac{d}{dt}M_x(t)\big|_{t=0} = e^{\lambda(e^t-1)}\lambda e^t\Big|_{t=0} = \lambda$.

$\mathrm{E}X^2 = \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)}\lambda e^t + \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda^2+\lambda.$

$\mathrm{Var}X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda.$

b.

$$
\begin{aligned}
M_x(t) &= \sum_{x=0}^{\infty} e^{tx}p(1-p)^x \quad = \quad p\sum_{x=0}^{\infty}((1-p)e^t)^x \\
&= p\frac{1}{1-(1-p)e^t} \quad = \quad \frac{p}{1-(1-p)e^t}, \quad t < -\log(1-p). \\
\mathrm{E}X &= \frac{d}{dt}M_x(t)\Big|_{t=0} \quad = \quad \frac{-p}{(1-(1-p)e^t)^2}\left(-(1-p)e^t\right)\Big|_{t=0} \\
&= \frac{p(1-p)}{p^2} \quad = \quad \frac{1-p}{p}. \\
\mathrm{E}X^2 &= \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} \\
&= \frac{\left(1-(1-p)e^t\right)^2\left(p(1-p)e^t\right) + p(1-p)e^t 2\left(1-(1-p)e^t\right)(1-p)e^t}{(1-(1-p)e^t)^4}\Bigg|_{t=0} \\
&= \frac{p^3(1-p)+2p^2(1-p)^2}{p^4} \quad = \quad \frac{p(1-p)+2(1-p)^2}{p^2}. \\
\mathrm{Var}X &= \frac{p(1-p)+2(1-p)^2}{p^2} - \frac{(1-p)^2}{p^2} \quad = \quad \frac{1-p}{p^2}.
\end{aligned}
$$

c. $M_x(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x^2-2\mu x-2\sigma^2 tx+\mu^2)/2\sigma^2}dx$. Now complete the square in the numerator by writing

$$
\begin{aligned}
x^2-2\mu x-2\sigma^2 tx+\mu^2 &= x^2-2(\mu+\sigma^2 t)x \pm (\mu+\sigma^2 t)^2+\mu^2 \\
&= (x-(\mu+\sigma^2 t))^2-(\mu+\sigma^2 t)^2+\mu^2 \\
&= (x-(\mu+\sigma^2 t))^2-[2\mu\sigma^2 t+(\sigma^2 t)^2].
\end{aligned}
$$

Then we have $M_x(t) = e^{[2\mu\sigma^2 t+(\sigma^2 t)^2]/2\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2}dx = e^{\mu t+\frac{\sigma^2 t^2}{2}}$.

$\mathrm{E}X = \frac{d}{dt}M_x(t)\big|_{t=0} = (\mu+\sigma^2 t)e^{\mu t+\sigma^2 t^2/2}\Big|_{t=0} = \mu.$

$\mathrm{E}X^2 = \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} = (\mu+\sigma^2 t)^2 e^{\mu t+\sigma^2 t^2/2}+\sigma^2 e^{\mu t+\sigma^2 t/2}\Big|_{t=0} = \mu^2+\sigma^2.$

$\mathrm{Var}X = \mu^2+\sigma^2-\mu^2 = \sigma^2.$

2.35 a.

$$
\begin{aligned}
\mathrm{E}X_1^r &= \int_0^{\infty} x^r \frac{1}{\sqrt{2\pi}x}e^{-(\log x)^2/2}dx && (f_1 \text{ is lognormal with } \mu=0,\ \sigma_2=1) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{y(r-1)}e^{-y^2/2}e^y dy && (\text{substitute } y=\log x,\ dy=(1/x)dx) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2+ry}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y^2-2ry+r^2)/2}e^{r^2/2}dy \\
&= e^{r^2/2}.
\end{aligned}
$$

b.

$$\begin{aligned}
\int_0^\infty x^r f_1(x)\sin(2\pi\log x)dx &= \int_0^\infty x^r \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}\sin(2\pi\log x)dx \\
&= \int_{-\infty}^\infty e^{(y+r)r}\frac{1}{\sqrt{2\pi}} e^{-(y+r)^2/2}\sin(2\pi y+2\pi r)dy \\
&\qquad\text{(substitute } y=\log x,\ dy=(1/x)dx) \\
&= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{(r^2-y^2)/2}\sin(2\pi y)dy \\
&\qquad(\sin(a+2\pi r)=\sin(a) \text{ if } r=0,1,2,\ldots) \\
&= 0,
\end{aligned}$$

because $e^{(r^2-y^2)/2}\sin(2\pi y) = -e^{(r^2-(-y)^2)/2}\sin(2\pi(-y))$; the integrand is an odd function so the negative integral cancels the positive one.

2.36 First, it can be shown that

$$\lim_{x\to\infty} e^{tx-(\log x)^2} = \infty$$

by using l'Hôpital's rule to show

$$\lim_{x\to\infty} \frac{tx-(\log x)^2}{tx} = 1,$$

and, hence,

$$\lim_{x\to\infty} tx-(\log x)^2 = \lim_{x\to\infty} tx = \infty.$$

Then for any $k>0$, there is a constant c such that

$$\int_k^\infty \frac{1}{x} e^{tx} e^{(\log x)^2/2}dx \geq c\int_k^\infty \frac{1}{x}dx = c\log x|_k^\infty = \infty.$$

Hence $M_x(t)$ does not exist.

2.37 a. The graph looks very similar to Figure 2.3.2 except that f_1 is symmetric around 0 (since it is standard normal).

b. The functions look like $t^2/2$ – it is impossible to see any difference.

c. The mgf of f_1 is $e^{K_1(t)}$. The mgf of f_2 is $e^{K_2(t)}$.

d. Make the transformation $y=e^x$ to get the densities in Example 2.3.10.

2.39 a. $\frac{d}{dx}\int_0^x e^{-\lambda t}dt = e^{-\lambda x}$. Verify

$$\frac{d}{dx}\left[\int_0^x e^{-\lambda t}dt\right] = \frac{d}{dx}\left[-\frac{1}{\lambda}e^{-\lambda t}\bigg|_0^x\right] = \frac{d}{dx}\left(-\frac{1}{\lambda}e^{-\lambda x}+\frac{1}{\lambda}\right) = e^{-\lambda x}.$$

b. $\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}dt = \int_0^\infty \frac{d}{d\lambda}e^{-\lambda t}dt = \int_0^\infty -te^{-\lambda t}dt = -\frac{\Gamma(2)}{\lambda^2} = -\frac{1}{\lambda^2}$. Verify

$$\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}dt = \frac{d}{d\lambda}\frac{1}{\lambda} = -\frac{1}{\lambda^2}.$$

c. $\frac{d}{dt}\int_t^1 \frac{1}{x^2}dx = -\frac{1}{t^2}$. Verify

$$\frac{d}{dt}\left[\int_t^1 \frac{1}{x^2}dx\right] = \frac{d}{dt}\left(-\frac{1}{x}\bigg|_t^1\right) = \frac{d}{dt}\left(-1+\frac{1}{t}\right) = -\frac{1}{t^2}.$$

d. $\frac{d}{dt}\int_1^\infty \frac{1}{(x-t)^2}dx = \int_1^\infty \frac{d}{dt}\left(\frac{1}{(x-t)^2}\right)dx = \int_1^\infty 2(x-t)^{-3}dx = -(x-t)^{-2}\Big|_1^\infty = \frac{1}{(1-t)^2}$. Verify

$$\frac{d}{dt}\int_1^\infty (x-t)^{-2}dx = \frac{d}{dt}\left[-(x-t)^{-1}\Big|_1^\infty\right] = \frac{d}{dt}\frac{1}{1-t} = \frac{1}{(1-t)^2}.$$

(2.13 ending)

Doing the summations:

Use the known moments of the geometric distribution.
From appendix, if $Y \sim \text{Geometric}(p)$,

then $$EY = \sum_{k=1}^{\infty} k\,p(1-p)^{k-1} = \frac{1}{p}.$$

This implies $$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}.$$

Replacing p by $1-p$ in this gives

$$\sum_{k=1}^{\infty} k\,p^{k-1} = \frac{1}{(1-p)^2}.$$

Plugging these values back into the expression for EX we get

$$EX = (1-p)p\left(\frac{1}{(1-p)^2} + \frac{1}{p^2}\right)$$

$$= \frac{p}{1-p} + \frac{1-p}{p}$$ same as before.

Problem 2.14(b)
MSims

Let X be a discrete random variable whose range is the non negative integers.

$$EX = \sum_{x=0}^{\infty} x\, f_X(x)$$

$$= \sum_{x=1}^{\infty} x\, f_X(x)$$

$$= f_X(1) + 2f_X(2) + 3f_X(3) + 4f_X(4) + \dots$$

$$\begin{aligned} = f_X(1) + f_X(2) + f_X(3) + f_X(4) + \dots \\ + f_X(2) + f_X(3) + f_X(4) + \dots \\ + f_X(3) + f_X(4) + \dots \\ + f_X(4) + \dots \end{aligned}$$

$$= \sum_{k=1}^{\infty} f_X(k) + \sum_{k=2}^{\infty} f_X(k) + \sum_{k=3}^{\infty} f_X(k)$$

$$= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} f_X(k) \quad \text{but} \quad \sum_{k=j}^{\infty} f_X(k) = 1 - F_X(j-1)$$

$$= \sum_{j=1}^{\infty} [1 - F_X(j-1)] = \sum_{j=0}^{\infty} [1 - F_X(j)]$$

This is the discrete analogue to part (a).

[2.19] Assume EX^2 exists (is finite).
This implies EX exists.
Now note that

$$E(X-a)^2 = E(X^2-2aX+a^2)$$
$$= EX^2-2a(EX)+a^2.$$

Thus

$$\frac{d}{da}E(X-a)^2 = -2EX+2a \qquad (*)$$

and it is clear that

$$\frac{d}{da}E(X-a)^2 = 0 \text{ iff } EX = a.$$

From (*) we get

$$\frac{d^2}{da^2}E(X-a)^2 = 2 > 0.$$

Thus $a = EX$ is the unique value of a which minimizes $E(X-a)^2$.

The only assumption we needed is that EX^2 exists. If X is continuous with pdf $f_X(\cdot)$ this means that $\int_{-\infty}^{\infty} x^2 f_X(x)\,dx < \infty$.

a) The original function is, [2.22 ALTERNATE SOLUTION]

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} \qquad 0 < x < \infty, \quad \beta > 0$$

The goal is to transform to a gamma integral.

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}}\,dx$$

$$= \frac{2}{\beta^3\sqrt{\pi}} \int_0^\infty \cancel{x}\, e^{-\frac{x^2}{\beta^2}}\, 2x\,dx \qquad \text{Let } y = \frac{x^2}{\beta^2}$$

$$\beta^2 dy = 2x\,dx$$

$$= \frac{2}{\beta^3\sqrt{\pi}} \int_0^\infty \sqrt{y}\cdot\beta\cdot e^{-y}\cdot\beta^2 dy \qquad x = \sqrt{y}\,\beta$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{y}\, e^{-y}\,dy$$

$$= \frac{1}{\Gamma(\frac{3}{2})} \int_0^\infty y^{(\frac{3}{2}-1)} e^{-y}\,dy = \frac{\Gamma(3/2)}{\Gamma(3/2)} = 1$$

Recall: $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ so that

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

PROBLEM 2-22

b) Find $E(X)$ + $V(X)$

$$E(X) = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x \left(\frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} dx\right)$$

Let $y = \frac{x^2}{\beta^2}$

$\beta^2 dy = 2x\,dx$

$x = \sqrt{y}\,\beta$

$x^3 = y\sqrt{y}\,\beta^3$

$$= \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} y \cdot \sqrt{y} \cdot \beta^3 e^{-y} \frac{\beta^2 dy}{2\sqrt{y}\,\beta}$$

$$= \frac{2\beta}{\sqrt{\pi}} \int_0^\infty y e^{-y} dy = \frac{2\beta}{\sqrt{\pi}} \int_0^\infty y^{2-1} e^{-y} dy$$

$$= \frac{2\beta}{\sqrt{\pi}} \Gamma(2) = \frac{2\beta}{\sqrt{\pi}} \quad \text{since } \Gamma(2) = 1! = 1.$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^\infty x^2 f(x)\,dx$$

$$= \int_0^\infty x^4 \cdot \frac{4}{\beta^3\sqrt{\pi}} e^{\frac{-x^2}{\beta^2}} dx$$

Let $y = \frac{x^2}{\beta^2}$

$\beta^2 dy = 2x\,dx$

$dx = \frac{\beta^2 dy}{2x}$

$x = \sqrt{y}\,\beta$

$x^4 = y^2\beta^4$

$$= \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} \cdot y^2\beta^4 \cdot e^{-y} \frac{\beta^2 dy}{2\beta\sqrt{y}}$$

$$= \frac{2\beta^2}{\sqrt{\pi}} \int_0^\infty \frac{y^2}{\sqrt{y}} e^{-y} dy$$

$$= \frac{2\beta^2}{\sqrt{\pi}} \int_0^\infty y\sqrt{y}\, e^{-y} dy$$

$$= \frac{2\beta^2}{\sqrt{\pi}} \underbrace{\int_0^\infty y^{\frac{3}{2}} e^{-y} dy}_{\text{gama integral } \Gamma(\frac{5}{2})}$$

$\Gamma(\frac{5}{2}) = \Gamma(\frac{3}{2}+1)$

$= \frac{3}{2}\Gamma(\frac{3}{2})$

$= \frac{3}{4}\sqrt{\pi}$

$$= \frac{2\beta^2}{\sqrt{\pi}} \cdot \frac{3}{4}\sqrt{\pi}$$

$$= \frac{3}{2}\beta^2$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{3}{2}\beta^2 - \left(\frac{2\beta}{\sqrt{\pi}}\right)^2 = \frac{3\pi\beta^2 - 8\beta^2}{2\pi} = \frac{\beta^2(3\pi - 8)}{2\pi}$$

Problem 2.38

MSims

Let X have a negative binomial distribution with pmf

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad \begin{array}{l} x = 0, 1, 2, \ldots \\ 0 < p < 1 \\ r > 1 \text{ an integer} \end{array}.$$

a) Calculate the mgf of X.

$$M_X(t) = E e^{tX}$$
$$= \sum_{x=0}^{\infty} e^{tx} P(X=x)$$
$$= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x$$
$$= p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} ((1-p)e^t)^x$$
$$= \left[\frac{p}{1-e^t(1-p)}\right]^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} (1-e^t(1-p))^r (e^t(1-p))^x$$
$$= \left[\frac{p}{1-e^t(1-p)}\right]^r \quad \text{when } t < -\log(1-p)$$

Note that $\sum_{x=0}^{\infty} \binom{r+x-1}{x} (1-e^t(1-p))^r (e^t(1-p))^x$ is the sum $\sum_{x=0}^{\infty} P(Y=x) = 1$, $Y \sim \text{Neg Bin}(r, 1-e^t(1-p))$ as long as $e^t(1-p) < 1$ $(t < -\log(1-p))$.

Problem 2.38 MS Ins

b) Define $Y = 2pX$

$$M_Y(t) = M_{2pX}(t) = M_X(2pt)$$

$$= \left[\frac{p}{1 - e^{2pt}(1-p)}\right]^r \quad \text{for } t < \frac{-\log(1-p)}{2p}$$

$$\lim_{p \to 0} M_Y(t) = \lim_{p \to 0} \left[\frac{p}{1 - e^{2pt}(1-p)}\right]^r$$

$$= \lim_{p \to 0} \left(\frac{1}{e^{2pt}}\right)^r \left(\frac{p}{e^{-2pt} - 1 + p}\right)^r$$

$$= 1 \left(\frac{\lim_{p \to 0} 1}{\lim_{p \to 0} -2te^{-2pt} + 1}\right)^r \quad \text{(L'Hopital)}$$

$$= \left(\frac{1}{1-2t}\right)^r$$

and as $p \to 0$ $t < \frac{-\log(1-p)}{2p}$ implies $t < \frac{1}{2}$

so $\lim_{p \to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^r$ for $t < \frac{1}{2}$

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2.40 Prove $\sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} = (n-x)\binom{n}{x}\int_0^{1-p} t^{n-x-1}(1-t)^x \, dt.$

$$\frac{d}{dp}\left[\sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right]$$

$$= \frac{d}{dp}\left[(1-p)^n + \sum_{k=1}^{x}\binom{n}{k}p^k(1-p)^{n-k}\right]$$

$$= -n(1-p)^{n-1} + \sum_{k=1}^{x}\left[\binom{n}{k}k\,p^{k-1}(1-p)^{n-k} - \binom{n}{k}(n-k)p^k(1-p)^{n-k-1}\right]$$

$$= -n(1-p)^{n-1} + \sum_{k=1}^{x}\left[\frac{n!}{(k-1)!\,(n-k)!}p^{k-1}(1-p)^{n-k} - \frac{n!}{k!\,(n-k-1)!}p^k(1-p)^{n-k-1}\right]$$

$$\left(\text{by letting } f(k) = \frac{n!}{(k-1)!\,(n-k)!}p^{k-1}(1-p)^{n-k}\right)$$

$$= -n(1-p)^{n-1} + \sum_{k=1}^{x}\left[f(k) - f(k+1)\right]$$

$$= -n(1-p)^{n-1} + \left[f(1) - \cancel{f(2)} + \cancel{f(2)} - \cancel{f(3)} + \cdots + \cancel{f(x)} - f(x+1)\right]$$

$$= -n(1-p)^{n-1} + n(1-p)^{n-1} - \frac{n!}{x!\,(n-x-1)!}p^x(1-p)^{n-x-1}$$

$$= -(n-x)\binom{n}{x}p^x(1-p)^{n-x-1}.$$

$$\frac{d}{dp}\left[(n-x)\binom{n}{x}\int_0^{1-p} t^{n-x-1}(1-t)^x\,dt\right]$$

$$\left(\text{by letting } \int t^{n-x-1}(1-t)^x\,dt = F_1(t)\right)$$

$$= (n-x)\binom{n}{x}\frac{d}{dp}\left[F_1(1-p) - F_1(0)\right]$$

$$= (n-x)\binom{n}{x}\left[F_1'(1-p)\cdot\frac{d(1-p)}{dp} - 0\right]$$

$$= (n-x)\binom{n}{x}(-1)\cdot(1-p)^{n-x-1}(1-(1-p))^x$$

$$= -(n-x)\binom{n}{x}p^x(1-p)^{n-x-1}.$$

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2.40 (Continued)

$$\therefore \frac{d}{dp}\left[\sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \right] = \frac{d}{dp}\left[(n-x)\binom{n}{x} \int_0^{1-p} t^{n-x-1}(1-t)^x dt \right]$$

$$\therefore \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} = (n-x)\binom{n}{x} \int_0^{1-p} t^{n-x-1}(1-t)^x dt + C.$$

Let $p=1$.

$0 = 0 + C.$

$\therefore C = 0$.

$$\therefore \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} = (n-x)\binom{n}{x} \int_0^{1-p} t^{n-x-1}(1-t)^x dt.$$