Chapter 3

# Common Families of Distributions

3.1 The pmf of $X$ is $f(x) = \frac{1}{N_1-N_0+1}$, $x = N_0, N_0+1, \dots, N_1$. Then

$$
\begin{aligned}
\mathrm{E}X &= \sum_{x=N_0}^{N_1} x\frac{1}{N_1-N_0+1} = \frac{1}{N_1-N_0+1}\left(\sum_{x=1}^{N_1} x - \sum_{x=1}^{N_0-1} x\right) \\
&= \frac{1}{N_1-N_0+1}\left(\frac{N_1(N_1+1)}{2} - \frac{(N_0-1)(N_0-1+1)}{2}\right) \\
&= \frac{N_1+N_0}{2}.
\end{aligned}
$$

Similarly, using the formula for $\sum_1^N x^2$, we obtain

$$
\begin{aligned}
\mathrm{E}x^2 &= \frac{1}{N_1-N_0+1}\left(\frac{N_1(N_1+1)(2N_1+1) - N_0(N_0-1)(2N_0-1)}{6}\right) \\
\mathrm{Var}X &= \mathrm{E}X^2 - \mathrm{E}X = \frac{(N_1-N_0)(N_1-N_0+2)}{12}.
\end{aligned}
$$

3.2 Let $X$ = number of defective parts in the sample. Then $X \sim$ hypergeometric$(N = 100, M, K)$ where $M$ = number of defectives in the lot and $K$ = sample size.

a. If there are 6 or more defectives in the lot, then the probability that the lot is accepted $(X = 0)$ is at most

$$
P(X = 0 \mid M = 100, N = 6, K) = \frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}} = \frac{(100-K)\cdot\cdots\cdot(100-K-5)}{100\cdot\cdots\cdot 95}.
$$

By trial and error we find $P(X = 0) = .10056$ for $K = 31$ and $P(X = 0) = .09182$ for $K = 32$. So the sample size must be at least 32.

b. Now $P(\text{accept lot}) = P(X = 0 \text{ or } 1)$, and, for 6 or more defectives, the probability is at most

$$
P(X = 0 \text{ or } 1 \mid M = 100, N = 6, K) = \frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}} + \frac{\binom{6}{1}\binom{94}{K-1}}{\binom{100}{K}}.
$$

By trial and error we find $P(X = 0 \text{ or } 1) = .10220$ for $K = 50$ and $P(X = 0 \text{ or } 1) = .09331$ for $K = 51$. So the sample size must be at least 51.

Wrong

3.3 In the seven seconds for the event, no car must pass in the last three seconds, an event with probability $(1-p)^3$. The only occurrence in the first four seconds, for which the pedestrian does not wait the entire four seconds, is to have a car pass in the first second and no other car pass. This has probability $p(1-p)^3$. Thus the probability of waiting exactly four seconds before starting to cross is $[1 - p(1-p)^3](1-p)^3$.

3.5 Let $X$ = number of effective cases. If the new and old drugs are equally effective, then the probability that the new drug is effective on a case is .8. If the cases are independent then $X \sim$ binomial(100, .8), and

$$P(X \geq 85) = \sum_{x=85}^{100} \binom{100}{x} .8^x .2^{100-x} = .1285.$$

So, even if the new drug is no better than the old, the chance of 85 or more effective cases is not too small. Hence, we cannot conclude the new drug is better. Note that using a normal approximation to calculate this binomial probability yields $P(X \geq 85) \approx P(Z \geq 1.125) = .1303$.

3.7 Let $X \sim$ Poisson($\lambda$). We want $P(X \geq 2) \geq .99$, that is,

$$P(X \leq 1) = e^{-\lambda} + \lambda e^{-\lambda} \leq .01.$$

Solving $e^{-\lambda} + \lambda e^{-\lambda} = .01$ by trial and error (numerical bisection method) yields $\lambda = 6.6384$.

3.8 a. We want $P(X > N) < .01$ where $X \sim$ binomial(1000, 1/2). Since the 1000 customers choose randomly, we take $p = 1/2$. We thus require

$$P(X > N) = \sum_{x=N+1}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{1000-x} < .01$$

which implies that

$$\left(\frac{1}{2}\right)^{1000} \sum_{x=N+1}^{1000} \binom{1000}{x} < .01.$$

This last inequality can be used to solve for $N$, that is, $N$ is the smallest integer that satisfies

$$\left(\frac{1}{2}\right)^{1000} \sum_{x=N+1}^{1000} \binom{1000}{x} < .01.$$

The solution is $N = 537$.

b. To use the normal approximation we take $X \sim \text{n}(500, 250)$, where we used $\mu = 1000(\frac{1}{2}) = 500$ and $\sigma^2 = 1000(\frac{1}{2})(\frac{1}{2}) = 250$.Then

$$P(X > N) = P\left(\frac{X - 500}{\sqrt{250}} > \frac{N - 500}{\sqrt{250}}\right) < .01$$

thus,

$$P\left(Z > \frac{N - 500}{\sqrt{250}}\right) < .01$$

where $Z \sim \text{n}(0, 1)$. From the normal table we get

$$P(Z > 2.33) \approx .0099 < .01 \quad \Rightarrow \quad \frac{N - 500}{\sqrt{250}} = 2.33$$
$$\Rightarrow \quad N \approx 537.$$

Therefore, each theater should have at least 537 seats, and the answer based on the approximation equals the exact answer.

3.9 a. We can think of each one of the 60 children entering kindergarten as 60 independent Bernoulli trials with probability of success (a twin birth) of approximately $\frac{1}{90}$. The probability of having 5 or more successes approximates the probability of having 5 or more sets of twins entering kindergarten. Then $X \sim \text{binomial}(60, \frac{1}{90})$ and

$$P(X \geq 5) = 1 - \sum_{x=0}^{4} \binom{60}{x} \left(\frac{1}{90}\right)^x \left(1 - \frac{1}{90}\right)^{60-x} = .0006,$$

which is small and may be rare enough to be newsworthy.

b. Let $X$ be the number of elementary schools in New York state that have 5 or more sets of twins entering kindergarten. Then the probability of interest is $P(X \geq 1)$ where $X \sim \text{binomial}(310,.0006)$. Therefore $P(X \geq 1) = 1 - P(X = 0) = .1698$.

c. Let $X$ be the number of States that have 5 or more sets of twins entering kindergarten during any of the last ten years. Then the probability of interest is $P(X \geq 1)$ where $X \sim \text{binomial}(500, .1698)$. Therefore $P(X \geq 1) = 1 - P(X = 0) = 1 - 3.90 \times 10^{-41} \approx 1$.

3.11 a.

$$\begin{aligned}
&\lim_{M/N\to p, M\to\infty, N\to\infty} \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \\
&= \frac{K!}{x!(K-x)!} \lim_{M/N\to p, M\to\infty, N\to\infty} \frac{M!(N-M)!(N-K)!}{N!(M-x)!(N-M-(K-x))!}
\end{aligned}$$

In the limit, each of the factorial terms can be replaced by the approximation from Stirling's formula because, for example,

$$M! = (M!/(\sqrt{2\pi}M^{M+1/2}e^{-M}))\sqrt{2\pi}M^{M+1/2}e^{-M}$$

and $M!/(\sqrt{2\pi}M^{M+1/2}e^{-M}) \to 1$. When this replacement is made, all the $\sqrt{2\pi}$ and exponential terms cancel. Thus,

$$\begin{aligned}
&\lim_{M/N\to p, M\to\infty, N\to\infty} \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \\
&= \binom{K}{x} \lim_{M/N\to p, M\to\infty, N\to\infty} \frac{M^{M+1/2}(N-M)^{N-M+1/2}(N-K)^{N-K+1/2}}{N^{N+1/2}(M-x)^{M-x+1/2}(N-M-K+x)^{N-M-(K-x)+1/2}}.
\end{aligned}$$

We can evaluate the limit by breaking the ratio into seven terms, each of which has a finite limit we can evaluate. In some limits we use the fact that $M \to \infty$, $N \to \infty$ and $M/N \to p$ imply $N - M \to \infty$. The first term (of the seven terms) is

$$\lim_{M\to\infty} \left(\frac{M}{M-x}\right)^M = \lim_{M\to\infty} \frac{1}{\left(\frac{M-x}{M}\right)^M} = \lim_{M\to\infty} \frac{1}{\left(1+\frac{-x}{M}\right)^M} = \frac{1}{e^{-x}} = e^x.$$

Lemma 2.3.14 is used to get the penultimate equality. Similarly we get two more terms,

$$\lim_{N-M\to\infty} \left(\frac{N-M}{N-M-(K-x)}\right)^{N-M} = e^{K-x}$$

and

$$\lim_{N\to\infty} \left(\frac{N-K}{N}\right)^N = e^{-K}.$$

Note, the product of these three limits is one. Three other terms are

$$\begin{aligned} \lim M \to \infty \left(\frac{M}{M-x}\right)^{1/2} &= 1 \\ \lim_{N-M\to\infty} \left(\frac{N-M}{N-M-(K-x)}\right)^{1/2} &= 1 \end{aligned}$$

and

$$\lim_{N\to\infty} \left(\frac{N-K}{N}\right)^{1/2} = 1.$$

The only term left is

$$\begin{aligned} &\lim_{M/N\to p, M\to\infty, N\to\infty} \frac{(M-x)^x(N-M-(K-x))^{K-x}}{(N-K)^K} \\ &= \lim_{M/N\to p, M\to\infty, N\to\infty} \left(\frac{M-x}{N-K}\right)^x \left(\frac{N-M-(K-x)}{N-K}\right)^{K-x} \\ &= p^x(1-p)^{K-x}. \end{aligned}$$

b. If in (a) we in addition have $K \to \infty$, $p \to 0$, $MK/N \to pK \to \lambda$, by the Poisson approximation to the binomial, we heuristically get

$$\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \to \binom{K}{x} p^x(1-p)^{K-x} \to \frac{e^{-\lambda}\lambda^x}{x!}.$$

c. Using Stirling's formula as in (a), we get

$$\begin{aligned} &\lim_{N,M,K\to\infty, \frac{M}{N}\to 0, \frac{KM}{N}\to\lambda} \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \\ &= \lim_{N,M,K\to\infty, \frac{M}{N}\to 0, \frac{KM}{N}\to\lambda} \frac{e^{-x}}{x!} \frac{K^x e^x M^x e^x (N-M)^{K-x} e^{K-x}}{N^K e^K} \\ &= \frac{1}{x!} \lim_{N,M,K\to\infty, \frac{M}{N}\to 0, \frac{KM}{N}\to\lambda} \left(\frac{KM}{N}\right)^x \left(\frac{N-M}{N}\right)^{K-x} \\ &= \frac{1}{x!}\lambda^x \lim_{N,M,K\to\infty, \frac{M}{N}\to 0, \frac{KM}{N}\to\lambda} \left(1 - \frac{\frac{MK}{N}}{K}\right)^K \\ &= \frac{e^{-\lambda}\lambda^x}{x!}. \end{aligned}$$

3.12 Consider a sequence of Bernoulli trials with success probability $p$. Define $X$ = number of successes in first $n$ trials and $Y$ = number of failures before the $r$th success. Then $X$ and $Y$ have the specified binomial and hypergeometric distributions, respectively. And we have

$$\begin{aligned} F_x(r-1) &= P(X \le r-1) \\ &= P(r\text{th success on } (n+1)\text{st or later trial}) \\ &= P(\text{at least } n+1-r \text{ failures before the } r\text{th success}) \\ &= P(Y \ge n-r+1) \\ &= 1 - P(Y \le n-r) \\ &= 1 - F_Y(n-r). \end{aligned}$$

3.13 For any $X$ with support $0,1,\ldots$, we have the mean and variance of the $0-$*truncated* $X_T$ are given by

$$\begin{aligned}
\mathrm{E}X_T &= \sum_{x=1}^{\infty} xP(X_T=x) = \sum_{x=1}^{\infty} x\frac{P(X=x)}{P(X>0)}\\
&= \frac{1}{P(X>0)}\sum_{x=1}^{\infty} xP(X=x) = \frac{1}{P(X>0)}\sum_{x=0}^{\infty} xP(X=x) = \frac{\mathrm{E}X}{P(X>0)}.
\end{aligned}$$

In a similar way we get $\mathrm{E}X_T^2 = \frac{\mathrm{E}X^2}{P(X>0)}$. Thus,

$$\mathrm{Var}X_T = \frac{\mathrm{E}X^2}{P(X>0)} - \left(\frac{\mathrm{E}X}{P(X>0)}\right)^2.$$

a. For Poisson$(\lambda)$, $P(X>0) = 1-P(X=0) = 1-\frac{e^{-\lambda}\lambda^0}{0!} = 1-e^{-\lambda}$, therefore

$$\begin{aligned}
P(X_T=x) &= \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})} \quad x=1,2,\ldots\\
\mathrm{E}X_T &= \lambda/(1-e^{-\lambda})\\
\mathrm{Var}X_T &= (\lambda^2+\lambda)/(1-e^{-\lambda}) - (\lambda/(1-e^{-\lambda}))^2.
\end{aligned}$$

b. For negative binomial$(r,p)$, $P(X>0) = 1-P(X=0) = 1-\binom{r-1}{0}p^r(1-p)^0 = 1-p^r$. Then

$$\begin{aligned}
P(X_T=x) &= \frac{\binom{r+x-1}{x}p^r(1-p)^x}{1-p^r}, \quad x=1,2,\ldots\\
\mathrm{E}X_T &= \frac{r(1-p)}{p(1-p^r)}\\
\mathrm{Var}X_T &= \frac{r(1-p)+r^2(1-p)^2}{p^2(1-p^r)} - \left[\frac{r(1-p)}{p(1-p^r)^2}\right].
\end{aligned}$$

3.14 a. $\sum_{x=1}^{\infty}\frac{-(1-p)^x}{x\log p} = \frac{1}{\log p}\sum_{x=1}^{\infty}\frac{-(1-p)^x}{x} = 1$, since the sum is the Taylor series for $\log p$.

b.

$$\mathrm{E}X = \frac{-1}{\log p}\left[\sum_{x=1}^{\infty}(1-p)^x\right] = \frac{-1}{\log p}\left[\sum_{x=0}^{\infty}(1-p)^x - 1\right] == \frac{-1}{\log p}\left[\frac{1}{p}-1\right] = \frac{-1}{\log p}\left(\frac{1-p}{p}\right).$$

Since the geometric series converges uniformly,

$$\begin{aligned}
\mathrm{E}X^2 &= \frac{-1}{\log p}\sum_{x=1}^{\infty} x(1-p)^x = \frac{(1-p)}{\log p}\sum_{x=1}^{\infty}\frac{d}{dp}(1-p)^x\\
&= \frac{(1-p)}{\log p}\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x = \frac{(1-p)}{\log p}\frac{d}{dp}\left[\frac{1-p}{p}\right] = \frac{-(1-p)}{p^2\log p}.
\end{aligned}$$

Thus

$$\mathrm{Var}X = \frac{-(1-p)}{p^2\log p}\left[1+\frac{(1-p)}{\log p}\right].$$

Alternatively, the mgf can be calculated,

$$M_x(t) = \frac{-1}{\log p}\sum_{x=1}^{\infty}\left[(1-p)e^t\right]^x = \frac{\log(1+pe^t-e^t)}{\log p}$$

and can be differentiated to obtain the moments.

3.15 The moment generating function for the negative binomial is

$$M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r = \left(1 + \frac{1}{r}\frac{r(1-p)(e^t-1)}{1-(1-p)e^t}\right)^r,$$

the term

$$\frac{r(1-p)(e^t-1)}{1-(1-p)e^t} \to \frac{\lambda(e^t-1)}{1} = \lambda(e^t-1) \quad \text{as } r \to \infty,\ p \to 1 \text{ and } r(p-1) \to \lambda.$$

Thus by Lemma 2.3.14, the negative binomial moment generating function converges to $e^{\lambda(e^t-1)}$, the Poisson moment generating function.

3.16 a. Using integration by parts with, $u = t^\alpha$ and $dv = e^{-t}dt$, we obtain

$$\Gamma(\alpha+1) = \int_0^\infty t^{(\alpha+1)-1}e^{-t}dt = t^\alpha(-e^{-t})\Big|_0^\infty - \int_0^\infty \alpha t^{\alpha-1}(-e^{-t})dt = 0 + \alpha\Gamma(\alpha) = \alpha\Gamma(\alpha).$$

b. Making the change of variable $z = \sqrt{2t}$, i.e., $t = z^2/2$, we obtain

$$\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}dt = \int_0^\infty \frac{\sqrt{2}}{z}e^{-z^2/2}zdz = \sqrt{2}\int_0^\infty e^{-z^2/2}dz = \sqrt{2}\frac{\sqrt{\pi}}{\sqrt{2}} = \sqrt{\pi}.$$

where the penultimate equality uses (3.3.14).

3.17

$$\begin{aligned} \mathrm{E}X^\nu &= \int_0^\infty x^\nu \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{(\nu+\alpha)-1}e^{-x/\beta}dx \\ &= \frac{\Gamma(\nu+\alpha)\beta^{\nu+\alpha}}{\Gamma(\alpha)\beta^\alpha} = \frac{\beta^\nu\Gamma(\nu+\alpha)}{\Gamma(\alpha)}. \end{aligned}$$

Note, this formula is valid for all $\nu > -\alpha$. The expectation does not exist for $\nu \leq -\alpha$.

3.18 If $Y \sim$ negative binomial$(r,p)$, its moment generating function is $M_Y(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, and, from Theorem 2.3.15, $M_{pY}(t) = \left(\frac{p}{1-(1-p)e^{pt}}\right)^r$. Now use L'Hôpital's rule to calculate

$$\lim_{p\to 0}\left(\frac{p}{1-(1-p)e^{pt}}\right) = \lim_{p\to 0}\frac{1}{(p-1)te^{pt}+e^{pt}} = \frac{1}{1-t},$$

so the moment generating function converges to $(1-t)^{-r}$, the moment generating function of a gamma$(r,1)$.

3.19 Repeatedly apply the integration-by-parts formula

$$\frac{1}{\Gamma(n)}\int_x^\infty z^{n-1}z^{-z}dz = \frac{x^{n-1}e^{-x}}{(n-1)!} + \frac{1}{\Gamma(n-1)}\int_x^\infty z^{n-2}z^{-z}dz,$$

until the exponent on the second integral is zero. This will establish the formula. If $X \sim$ gamma$(\alpha,1)$ and $Y \sim$ Poisson$(x)$. The probabilistic relationship is $P(X \geq x) = P(Y \leq \alpha-1)$.

3.21 The moment generating function would be defined by $\frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{tx}}{1+x^2}dx$. On $(0,\infty)$, $e^{tx} > x$, hence

$$\int_0^\infty \frac{e^{tx}}{1+x^2}dx > \int_0^\infty \frac{x}{1+x^2}dx = \infty,$$

thus the moment generating function does not exist.

3.22 a.

$$\begin{aligned}
\mathrm{E}(X(X-1)) &= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \quad (\text{let } y = x-2) \\
&= e^{-\lambda}\lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \;=\; e^{-\lambda}\lambda^2 e^{\lambda} \;=\; \lambda^2 \\
\mathrm{E}X^2 &= \lambda^2 + \mathrm{E}X \;=\; \lambda^2+\lambda \\
\mathrm{Var}X &= \mathrm{E}X^2 - (\mathrm{E}X)^2 \;=\; \lambda^2+\lambda-\lambda^2 \;=\; \lambda.
\end{aligned}$$

b.

$$\begin{aligned}
\mathrm{E}(X(X-1)) &= \sum_{x=0}^{\infty} x(x-1)\binom{r+x-1}{x} pr(1-p)^x \\
&= \sum_{x=2}^{\infty} r(r+1)\binom{r+x-1}{x-2} pr(1-p)^x \\
&= r(r+1)\frac{(1-p)^2}{p^2}\sum_{y=0}^{\infty}\binom{r+2+y-1}{y} pr+2(1-p)^y \\
&= r(r-1)\frac{(1-p)^2}{p^2},
\end{aligned}$$

where in the second equality we substituted $y = x-2$, and in the third equality we use the fact that we are summing over a negative binomial$(r+2,p)$ pmf. Thus,

$$\begin{aligned}
\mathrm{Var}X &= \mathrm{E}X(X-1) + \mathrm{E}X - (\mathrm{E}X)^2 \\
&= r(r+1)\frac{(1-p)^2}{p^2} + \frac{r(1-p)}{p} - \frac{r^2(1-p)^2}{p^2} \\
&= \frac{r(1-p)}{p^2}.
\end{aligned}$$

c.

$$\begin{aligned}
\mathrm{E}X^2 &= \int_0^\infty x^2 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta}dx \;=\; \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha+1}e^{-x/\beta}dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+2)\beta^{\alpha+2} \;=\; \alpha(\alpha+1)\beta^2. \\
\mathrm{Var}X &= \mathrm{E}X^2 - (\mathrm{E}X)^2 \;=\; \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 \;=\; \alpha\beta^2.
\end{aligned}$$

d. (Use 3.3.18)

$$\begin{aligned}
\mathrm{E}X &= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha)} \;=\; \frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)} \;=\; \frac{\alpha}{\alpha+\beta}. \\
\mathrm{E}X^2 &= \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha)} \;=\; \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)} \;=\; \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}. \\
\mathrm{Var}X &= \mathrm{E}X^2 - (\mathrm{E}X)^2 \;=\; \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} \;=\; \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.
\end{aligned}$$

e. The double exponential$(\mu,\sigma)$ pdf is symmetric about $\mu$. Thus, by Exercise 2.26, $\mathrm{E}X=\mu$.

$$\begin{aligned}\mathrm{Var}X &= \int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{2\sigma}e^{-|x-\mu|/\sigma}dx \;=\; \int_{-\infty}^{\infty}\sigma z^2\frac{1}{2}e^{-|z|}\sigma dz\\ &= \sigma^2\int_0^{\infty}z^2e^{-z}dz \;=\; \sigma^2\Gamma(3) \;=\; 2\sigma^2.\end{aligned}$$

3.23 a.

$$\int_{\alpha}^{\infty}x^{-\beta-1}dx=\frac{-1}{\beta}x^{-\beta}\bigg|_{\alpha}^{\infty}=\frac{1}{\beta\alpha^{\beta}},$$

thus $f(x)$ integrates to 1 .

b. $\mathrm{E}X^n=\frac{\beta\alpha^n}{(n-\beta)}$, therefore

$$\begin{aligned}\mathrm{E}X &= \frac{\alpha\beta}{(1-\beta)}\\ \mathrm{E}X^2 &= \frac{\alpha\beta^2}{(2-\beta)}\\ \mathrm{Var}X &= \frac{\alpha\beta^2}{2-\beta}-\frac{(\alpha\beta)^2}{(1-\beta)^2}\end{aligned}$$

c. If $\beta<2$ the integral of the second moment is infinite.

3.24 a. $f_x(x)=\frac{1}{\beta}e^{-x/\beta}$, $x>0$. For $Y=X^{1/\gamma}$, $f_Y(y)=\frac{\gamma}{\beta}e^{-y^{\gamma}/\beta}y^{\gamma-1}$, $y>0$. Using the transformation $z=y^{\gamma}/\beta$, we calculate

$$\mathrm{E}Y^n=\frac{\gamma}{\beta}\int_0^{\infty}y^{\gamma+n-1}e^{-y^{\gamma}/\beta}dy=\beta^{n/\gamma}\int_0^{\infty}z^{n/\gamma}e^{-z}dz=\beta^{n/\gamma}\Gamma\left(\frac{n}{\gamma}+1\right).$$

Thus $\mathrm{E}Y=\beta^{1/\gamma}\Gamma(\frac{1}{\gamma}+1)$ and $\mathrm{Var}Y=\beta^{2/\gamma}\left[\Gamma\left(\frac{2}{\gamma}+1\right)-\Gamma^2\left(\frac{1}{\gamma}+1\right)\right]$.

b. $f_x(x)=\frac{1}{\beta}e^{-x/\beta}$, $x>0$. For $Y=(2X/\beta)^{1/2}$, $f_Y(y)=ye^{-y^2/2}$, $y>0$ . We now notice that

$$\mathrm{E}Y=\int_0^{\infty}y^2e^{-y^2/2}dy=\frac{\sqrt{2\pi}}{2}$$

since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^2e^{-y^2/2}=1$, the variance of a standard normal, and the integrand is symmetric. Use integration-by-parts to calculate the second moment

$$\mathrm{E}Y^2=\int_0^{\infty}y^3e^{-y^2/2}dy=2\int_0^{\infty}ye^{-y^2/2}dy=2,$$

where we take $u=y^2$, $dv=ye^{-y^2/2}$. Thus $\mathrm{Var}Y=2(1-\pi/4)$.

c. The gamma$(a,b)$ density is

$$f_X(x)=\frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}.$$

Make the transformation $y=1/x$ with $dx=-dy/y^2$ to get

$$f_Y(y)=f_X(1/y)|1/y^2|=\frac{1}{\Gamma(a)b^a}\left(\frac{1}{y}\right)^{a+1}e^{-1/by}.$$

The first two moments are

$$
\begin{aligned}
\mathrm{E}Y &= \frac{1}{\Gamma(a)b^a}\int_0^\infty \left(\frac{1}{y}\right)^a e^{-1/by} = \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} = \frac{1}{(a-1)b} \\
\mathrm{E}Y^2 &= \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} = \frac{1}{(a-1)(a-2)b^2},
\end{aligned}
$$

and so $\mathrm{Var}Y = \frac{1}{(a-1)^2(a-2)b^2}$.

d. $f_x(x) = \frac{1}{\Gamma(3/2)\beta^{3/2}}x^{3/2-1}e^{-x/\beta}$, $x > 0$. For $Y = (X/\beta)^{1/2}$, $f_Y(y) = \frac{2}{\Gamma(3/2)}y^2e^{-y^2}$, $y > 0$. To calculate the moments we use integration-by-parts with $u = y^2$, $dv = ye^{-y^2}$ to obtain

$$
\mathrm{E}Y = \frac{2}{\Gamma(3/2)}\int_0^\infty y^3e^{-y^2}dy = \frac{2}{\Gamma(3/2)}\int_0^\infty ye^{-y^2}dy = \frac{1}{\Gamma(3/2)}
$$

and with $u = y^3, dv = ye^{-y^2}$ to obtain

$$
\mathrm{E}Y^2 = \frac{2}{\Gamma(3/2)}\int_0^\infty y^4e^{-y^2}dy = \frac{3}{\Gamma(3/2)}\int_0^\infty y^2e^{-y^2}dy = \frac{3}{\Gamma(3/2)}\sqrt{\pi}.
$$

Using the fact that $\frac{1}{2\sqrt{\pi}}\int_{-\infty}^\infty y^2e^{-y^2} = 1$, since it is the variance of a $\mathrm{n}(0,2)$, symmetry yields $\int_0^\infty y^2e^{-y^2}dy = \sqrt{\pi}$. Thus, $\mathrm{Var}Y = 6 - 4/\pi$, using $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

e. $f_x(x) = e^{-x}$, $x > 0$. For $Y = \alpha - \gamma\log X$, $f_Y(y) = e^{-e^{\frac{\alpha-y}{\gamma}}}e^{\frac{\alpha-y}{\gamma}}\frac{1}{\gamma}$, $-\infty < y < \infty$. Calculation of $\mathrm{E}Y$ and $\mathrm{E}Y^2$ cannot be done in closed form. If we define

$$
I_1 = \int_0^\infty \log xe^{-x}dx, \qquad I_2 = \int_0^\infty (\log x)^2e^{-x}dx,
$$

then $\mathrm{E}Y = \mathrm{E}(\alpha - \gamma\log x) = \alpha - \gamma I_1$, and $\mathrm{E}Y^2 = \mathrm{E}(\alpha-\gamma\log x)^2 = \alpha^2 - 2\alpha\gamma I_1 + \gamma^2 I_2$.The constant $I_1 = .5772157$ is called Euler's constant.

3.25 Note that if T is continuous then,

$$
\begin{aligned}
P(t \le T \le t+\delta | t \le T) &= \frac{P(t \le T \le t+\delta, t \le T)}{P(t \le T)} \\
&= \frac{P(t \le T \le t+\delta)}{P(t \le T)} \\
&= \frac{F_T(t+\delta) - F_T(t)}{1-F_T(t)}.
\end{aligned}
$$

Therefore from the definition of derivative,

$$
h_T(t) = \frac{1}{1-F_T(t)} = \lim_{\delta\to 0}\frac{F_T(t+\delta) - F_T(t)}{\delta} = \frac{F_T'(t)}{1-F_T(t)} = \frac{f_T(t)}{1-F_T(t)}.
$$

Also,

$$
-\frac{d}{dt}(\log[1 - F_T(t)]) = -\frac{1}{1-F_T(t)}(-f_T(t)) = h_T(t).
$$

3.26 a. $f_T(t) = \frac{1}{\beta}e^{-t/\beta}$ and $F_T(t) = \int_0^t \frac{1}{\beta}e^{-x/\beta}dx = -\left.e^{-x/\beta}\right|_0^t = 1 - e^{-t/\beta}$. Thus,

$$
h_T(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{(1/\beta)e^{-t/\beta}}{1-(1-e^{-t/\beta})} = \frac{1}{\beta}.
$$

b. $f_T(t) = \frac{\gamma}{\beta} t^{\gamma-1} e^{-t^\gamma/\beta}, t \geq 0$ and $F_T(t) = \int_0^t \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta} dx = \int_0^{t^\gamma/\beta} e^{-u} du = -\left. e^{-u}\right|_0^{t^\gamma/\beta} = 1 - e^{-t^\gamma/\beta}$, where $u = x^{\gamma/\beta}$. Thus,

$$h_T(t) = \frac{(\gamma/\beta) t^{\gamma-1} e^{-t^\gamma/\beta}}{e^{-t^\gamma/\beta}} = \frac{\gamma}{\beta} t^{\gamma-1}.$$

c. $F_T(t) = \frac{1}{1+e^{-(t-\mu)/\beta}}$ and $f_T(t) = \frac{e^{-(t-\mu)/\beta}}{\left(1+e^{-(t-\mu)/\beta}\right)^2}$. Thus,

$$h_T(t) = \frac{1}{\beta} e^{-(t-\mu)/\beta \left(1+e^{-(t-\mu)/\beta}\right)^2} \frac{1}{\frac{e^{-(t-\mu)/\beta}}{1+e^{-(t-\mu)/\beta}}} = \frac{1}{\beta} F_T(t).$$

3.27 a. The uniform pdf satisfies the inequalities of Exercise 2.27, hence is unimodal.

b. For the gamma$(\alpha, \beta)$ pdf $f(x)$, ignoring constants, $\frac{d}{dx} f(x) = \frac{x^{\alpha-2} e^{-x/\beta}}{\beta} [\beta(\alpha-1) - x]$, which only has one sign change. Hence the pdf is unimodal with mode $\beta(\alpha - 1)$.

c. For the $\mathrm{n}(\mu, \sigma^2)$ pdf $f(x)$, ignoring constants, $\frac{d}{dx} f(x) = \frac{x-\mu}{\sigma^2} e^{-(-x/\beta)^2/2\sigma^2}$, which only has one sign change. Hence the pdf is unimodal with mode $\mu$.

d. For the beta$(\alpha, \beta)$ pdf $f(x)$, ignoring constants,

$$\frac{d}{dx} f(x) = x^{\alpha-2}(1-x)^{\beta-2} \left[(\alpha-1) - x(\alpha+\beta-2)\right],$$

which only has one sign change. Hence the pdf is unimodal with mode $\frac{\alpha-1}{\alpha+\beta-2}$.

3.28 a. (i) $\mu$ known,

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right),$$

$h(x) = 1, \quad c(\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} I_{(0,\infty)}(\sigma^2), \quad w_1(\sigma^2) = -\frac{1}{2\sigma^2}, \quad t_1(x) = (x-\mu)^2.$

(ii) $\sigma^2$ known,

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(\mu \frac{x}{\sigma^2}\right),$$

$h(x) = \exp\left(\frac{-x^2}{2\sigma^2}\right), \quad c(\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\mu^2}{2\sigma^2}\right), \quad w_1(\mu) = \mu, \quad t_1(x) = \frac{x}{\sigma^2}.$

b. (i) $\alpha$ known,

$$f(x|\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}},$$

$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \ x > 0, \quad c(\beta) = \frac{1}{\beta^\alpha}, \quad w_1(\beta) = \frac{1}{\beta}, \quad t_1(x) = -x.$

(ii) $\beta$ known,

$$f(x|\alpha) = e^{-x/\beta} \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp((\alpha-1)\log x),$$

$h(x) = e^{-x/\beta}, \ x > 0, \quad c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \quad w_1(\alpha) = \alpha - 1, \quad t_1(x) = \log x.$

(iii) $\alpha, \beta$ unknown,

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp((\alpha-1)\log x - \frac{x}{\beta}),$$

$h(x) = I_{\{x>0\}}(x), \quad c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}, \quad w_1(\alpha) = \alpha - 1, \quad t_1(x) = \log x,$
$w_2(\alpha,\beta) = -1/\beta, \quad t_2(x) = x.$

c. (i) $\alpha$ known, $h(x) = x^{\alpha-1} I_{[0,1]}(x), \quad c(\beta) = \frac{1}{B(\alpha,\beta)}, \quad w_1(\beta) = \beta - 1, \quad t_1(x) = \log(1-x).$

(ii) $\beta$ known, $h(x) = (1-x)^{\beta-1} I_{[0,1]}(x), \quad c(\alpha) = \frac{1}{B(\alpha,\beta)}, \quad w_1(x) = \alpha - 1, \quad t_1(x) = \log x.$

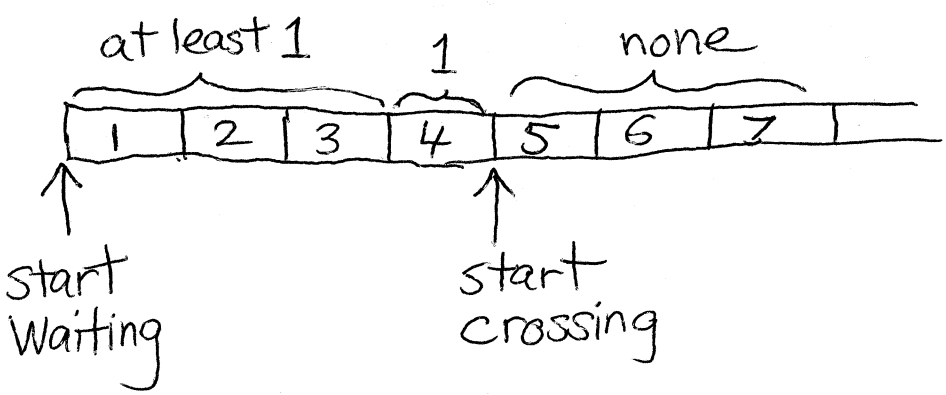
(3.3)

For the pedestrian to wait exactly 4 seconds, we require

(i) At least one car must pass in the first 3 seconds.

(ii) No cars must pass during seconds 5 through 7.

(iii) A car must pass during 4th second.




If condition (iii) is not true, the pedestrian could have crossed earlier.

Thus $P\{\text{wait exactly 4 seconds}\}$

$$= (1-(1-p)^3)\,p\,(1-p)^3.$$

# Exercises 3.4 and 3.6

**[3.4 (a)]** Here we are sampling from the keys with replacement so that each trial has probability $1/n$ of success. Thus the number of trials until the first success has a Geometric($p = 1/n$) distribution with mean $1/p = n$.

**[3.4(b)]** Now we are sampling with**out** replacement. This is equivalent to arranging the keys in a random order and trying them one by one. The correct key is equally likely to be in any of the positions $1, 2, \ldots, n$. Thus the number of keys we must try has a discrete uniform distribution on the numbers $1, 2, \ldots, n$ which has a mean of $(n + 1)/2$.

**[3.6]** Assuming the insects respond independently to the insecticide, the number of surviving insects $X$ has a Binomial($n = 2000, p = .01$) distribution. A reasonable approximation to this is the Poisson($\lambda = 2000 \times .01 = 20$) distribution. The Normal($\mu = np = 20, \sigma^2 = np(1 - p) = 19.8$) distribution (with a continuity correction) might also be fairly good.

The probability $P(X < 100)$ is just the sum over the Binomial($n = 2000, p = .01$) pmf from $x = 0$ to $x = 99$. The probability will be very close to 1. The normal approximation to this probability is $\Phi((99.5-20)/\sqrt{19.8}) = \Phi(17.8663) \approx 1$.

on next page

Addition to the Solution of 3.11(a)

The proof in the general case (in the handwritten solutions) is easier to understand if you first prove the special case where $x = 2$ and $K = 5$.

$$\begin{aligned}
& P(X = 2 \mid N, M, 5) \\
= \quad & \frac{\binom{M}{2}\binom{N-M}{3}}{\binom{N}{5}} \\
= \quad & \frac{\frac{M!}{2!(M-2)!} \cdot \frac{(N-M)!}{3!(N-M-3)!}}{\frac{N!}{5!(N-5)!}} \\
= \quad & \binom{5}{2} \frac{M(M-1) \cdot (N-M)(N-M-1)(N-M-2)}{N(N-1)(N-2)(N-3)(N-4)} \\
= \quad & \binom{5}{2} \frac{M(M-1)}{N(N-1)} \times \frac{(N-M)(N-M-1)(N-M-2)}{(N-2)(N-3)(N-4)} \\
\longrightarrow \quad & \binom{5}{2} (p \cdot p) \times ((1-p) \cdot (1-p) \cdot (1-p)) \\
= \quad & \binom{5}{2} p^2 (1-p)^3
\end{aligned}$$

since

$$\frac{M}{N}, \frac{M-1}{N-1} \longrightarrow p$$

and

$$\frac{N-M}{N-2}, \frac{N-M-1}{N-3}, \frac{N-M-2}{N-4} \longrightarrow 1-p$$

as $N \to \infty$, $M \to \infty$, and $M/N \to p$.

Yoonjung Lee

(3.11) Let $X$ have the hypergeometric distribution.

$$P(X=x \mid N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}, \quad x = 0, 1, 2, \dots K.$$

(a). Show that as $N \to \infty$, $M \to \infty$, and $M/N \to p$,

$$P(X=x \mid N, M, K) \to \binom{K}{x} p^x (1-p)^{K-x}. \quad x = 0, 1, 2, \dots K.$$

< solution >

$$P(X=x \mid N, M, K) = \frac{\dfrac{M!}{x!(M-x)!} \times \dfrac{(N-M)!}{(K-x)!(N-M-K+x)!}}{\dfrac{N!}{K!(N-K)!}}$$

$$= \frac{K!}{x!(K-x)!} \times \frac{M!}{(M-x)!} \times \frac{(N-M)!}{(N-M-K+x)!} \times \frac{(N-K)!}{N!}$$

$$= \binom{K}{x} \times \frac{M(M-1)\cdots(M-x+1)\cdot(N-M)(N-M-1)\cdots(N-M-K+x+1)}{N(N-1)(N-2)\cdots(N-K+1)!}$$

$$= \binom{K}{x} \times \prod_{i=1}^{x} \frac{M-(i-1)}{N-(i-1)} \times \prod_{i=1}^{K-x} \frac{N-M-(i-1)}{N-x-(i-1)}.$$

$$\frac{M-(i-1)}{N-(i-1)} \to p, \text{ as } N \to \infty,\ M \to \infty, \text{ and } M/N \to p. \quad (i = 1, 2, \dots x).$$

$$\frac{N-M-(i-1)}{N-K-(i-1)} = \frac{N}{N-K-(i-1)} - \frac{M+(i-1)}{N-K-(i-1)} \to 1-p, \text{ as } N \to \infty,\ M \to \infty, \text{ and } M/N \to p.$$

$(i = 1, 2, \dots K-x)$.

$$\therefore P(X=x \mid N, M, K) \to \binom{K}{x} p^x (1-p)^{K-x} \text{ as } N \to \infty,\ M \to \infty, \text{ and } M/N \to p.$$

[3.20]

(a) $$EX = \frac{2}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} \left( -e^{-x^2/2} \Big|_0^\infty = \frac{2}{\sqrt{2\pi}} \right.$$

$$EX^2 = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^2 e^{-x^2/2} dx \qquad (\dagger)$$

(since the integrand is symmetric about $x=0$)

$= 1$ since the integral above is the one you must compute to get the second moment of the $N(0,1)$ distribution which we know is 1 (use appendix).

Thus $\operatorname{Var} X = 1 - \left(\frac{2}{\sqrt{2\pi}}\right)^2 = 1 - \frac{2}{\pi}$

Alternative approaches to $EX^2$:

Integration by parts: (starting from $(\dagger)$)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^2 e^{-x^2/2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left( -x e^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right)$$

$$= 0 + \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{N(0,1) \text{ pdf}} dx = 1$$

Conversion to Gamma integral:

$$\frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-x^2/2} dx$$

Let $u = x^2/2$

$x = \sqrt{2}\sqrt{u}$

$dx = \frac{\sqrt{2}}{2} u^{-1/2}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2u e^{-u} \frac{\sqrt{2}}{2} u^{-1/2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} u^{3/2-1} e^{-u} du = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \text{ using } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$= 1 \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

[3.20] (b) Try $Y = g(X) = X^c$ with $c > 0$ and see what happens.

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \text{ for } y > 0$$

where $g^{-1}(y) = y^{1/c}$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{c} y^{(1/c)-1} > 0$$

$$= \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \Big|_{x = y^{1/c}} \cdot \frac{1}{c} y^{(1/c)-1}$$

$$= \frac{2}{\sqrt{2\pi}} e^{-y^{(2/c)}/2} \cdot \frac{1}{c} y^{(1/c)-1}$$

Taking $c = 2$ gives

$$= \frac{1}{\sqrt{2\pi}} y^{(1/2)-1} e^{-y/2} \text{ for } y > 0.$$

This is a Gamma$(\alpha = \frac{1}{2}, \beta = 2)$ pdf.

Note: $\frac{1}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{2^{1/2}\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}$

Thus $Y = X^2$ has the above Gamma distn.

[3.22(b)] $E\, X(X-1)$

$$= \sum_{x=0}^{\infty} x(x-1)\binom{r+x-1}{x} p^r (1-p)^x$$

$$= \sum_{x=2}^{\infty} \underbrace{x(x-1)\binom{r+x-1}{x}} p^r (1-p)^x$$

$$x(x-1)\frac{(r+x-1)!}{x!(r-1)!}$$

$$= r(r+1)\frac{(r+x-1)!}{(x-2)!(r+1)!}$$

$$= r(r+1)\binom{r+x-1}{x-2}$$

$$= r(r+1)\sum_{x=2}^{\infty}\binom{r+x-1}{x-2} p^r (1-p)^x$$

Let $y = x-2$, $x = y+2$.

$$= r(r+1)\sum_{y=0}^{\infty}\binom{r+2+y-1}{y} p^r (1-p)^{y+2}$$

$$= r(r+1)\frac{(1-p)^2}{p^2}\sum_{y=0}^{\infty}\underbrace{\binom{r+2+y-1}{y} p^{r+2}(1-p)^y}_{\text{NegBin}(r+2,\,p)\text{ pmf}}$$

$$= r(r+1)\frac{(1-p)^2}{p^2}$$

$$\operatorname{Var} X = E\,X(X-1) + EX - (EX)^2$$

$$= r(r+1)\frac{(1-p)^2}{p^2} + r\frac{(1-p)}{p} - r^2\frac{(1-p)^2}{p^2}$$

$$= \frac{r(1-p)}{p^2}\left[(r+1)(1-p) + p - r(1-p)\right]$$

$$= \frac{r(1-p)}{p^2}$$

Note: The variance can also be computed from the mgf. (But you must first compute the mgf!)

**3.27** Comment: A density $f(x)$ will be unimodal if its derivative $f'(x)$ has one sign change, going from $+$ to $-$. It will also be unimodal (with mode at the left endpoint of the support) if the derivative is always negative. It will be unimodal (with mode at the right endpoint of the support) if the derivative is always positive.

**3.27(b)** The solution in the manual is only correct for $\alpha > 1$. For $0 < \alpha \leq 1$, the density is unimodal with mode at zero.

**3.27(d)** The solution is correct if both $\alpha > 1$ and $\beta > 1$. If both $\alpha < 1$ and $\beta < 1$, then the density is **not** unimodal; the density has two peaks (at $x = 0$ and $x = 1$). If $\alpha < 1$ and $\beta \geq 1$, then it is unimodal with mode at $x = 0$. If $\alpha \geq 1$ and $\beta < 1$, then it is unimodal with mode at $x = 1$. If $\alpha = \beta = 1$, then the density is uniform; it is unimodal and any value $x \in [0, 1]$ can be taken as the mode.