Independent Random Variables

X, Y, Z are <u>mutually</u> <u>independent</u> rv's ...

if $P(X \in A, Y \in B, Z \in C) = P(X \in A)\, P(Y \in B)\, P(Z \in C)$
for all A, B, C.

or equivalently

if X, Y, Z have a joint pdf (or pmf)
$f_{X,Y,Z}(x,y,z)$ so that

$$P((X,Y,Z) \in D) = \iiint_D f_{X,Y,Z}(x,y,z)\, dx\, dy\, dz$$

for all regions D,

if $f_{X,Y,Z}(x,y,z) = f_X(x)\, f_Y(y)\, f_Z(z)$
for all x, y, z.

<u>Identically</u> <u>distributed</u> rv's have the same cdf.

X, Y, Z are <u>i.i.d</u> (independent and identically distributed)
if they are mutually independent and have the same distribution (cdf).

Various Facts

$$E(X+Y+Z) = EX + EY + EZ$$

If X, Y, Z are mutually indep., then

$$Var(X+Y+Z) = Var(X) + Var(Y) + Var(Z)$$

$$M_{X+Y+Z}(t) = M_X(t)\, M_Y(t)\, M_Z(t).$$

If $X_1, X_2, \ldots, X_n$ are i.i.d., then

$$E\left(\sum_{i=1}^{n} X_i\right) = n(EX_1)$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = n(Var\, X_1)$$

$$M_{\sum_{i=1}^{n} X_i}(t) = \left[M_{X_1}(t)\right]^n.$$

More Facts about Independent Random Variables

If X, Y, Z are mutually independent rv's:

- $E(XYZ) = EX \cdot EY \cdot EZ$
  (whenever EX, EY, EZ are finite)

- $g(X)$, $h(Y)$, $k(Z)$ are mutually independent rv's for any functions g, h, k,

  and therefore ...

- $Eg(X)h(Y)k(Z) = Eg(X)\, Eh(Y)\, Ek(Z)$
  (whenever $Eg(X)$, $Eh(Y)$, $Ek(Z)$ are finite),

  and therefore ...

- $M_{X+Y+Z}(t) = Ee^{t(X+Y+Z)} = Ee^{tX}e^{tY}e^{tZ}$
  $= Ee^{tX}\, Ee^{tY}\, Ee^{tZ} = M_X(t)M_Y(t)M_Z(t)$.

Discrete Distributions

Binomial
Geometric
Negative Binomial
Hypergeometric
Poisson

$\left.\begin{array}{l}\text{Binomial}\\ \text{Geometric}\\ \text{Negative Binomial}\end{array}\right\}$ arise from Bernoulli trials

Bernoulli trials

$$X_1, X_2, X_3, \ldots \text{ iid } \underbrace{\text{Bernoulli}(p)}$$

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

① $EX_i = p$

② $\operatorname{Var} X_i = p(1-p)$

③ $M_{X_i}(t) = 1 - p + pe^t$ for all t

Proofs: For convenience, let $X = X_1$.

① $EX = \sum_{x \in \mathcal{X}} x f_X(x) = 1 \cdot p + 0 \cdot (1-p) = p$

② $\mathrm{Var}\, X = EX^2 - \underbrace{(EX)^2}_{p^2}$

Since $X^2 = X$ [and $X^k = X$ for all k since X takes on only the values 0 and 1 for which $1^k = 1$ and $0^k = 0$.]

we have $EX^2 = p$.

Thus $\mathrm{Var}\, X = p - p^2 = p(1-p)$.

③ $M_X(t) = Ee^{tX} = \sum_{x \in \mathcal{X}} e^{tx} f_X(x)$

$= e^{t0} \cdot f_X(0) + e^{t1} \cdot f_X(1)$

$= 1 - p + pe^t$

Think of $X_1, X_2, X_3, \ldots$ as the results of repeated trials or tosses of a coin:

$X_i = I_{\{\text{success on trial } i\}}$

or $X_i = I_{\{\text{heads on } i^{th} \text{ toss}\}}$.

The Binomial distribution

Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli(p).

Define $S_n = \sum_{i=1}^{n} X_i$.

Then $S_n \sim$ Binomial(n, p)

with pmf

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n,$$

and

$$E\, S_n = n\, EX_1 = np,$$

$$\mathrm{Var}\, S_n = n\, \mathrm{Var}\, X_1 = np(1-p),$$

$$M_{S_n}(t) = [M_{X_1}(t)]^n = (1 - p + pe^t)^n.$$

(These results can also be obtained directly from the pmf.)

The Geometric distribution

Let $X_1, X_2, X_3, \ldots$ be iid Bernoulli(p).

Define T_1 = the # of the trial at which the first success ($=1$) occurs

$$= \inf\{n : X_n = 1\}.$$

$\inf$: smallest

Then $T_1 \sim$ Geometric(p),

$$P(T_1 = k) = (1-p)^{k-1} p \quad \text{for } k = 1, 2, 3, \ldots$$

$$E\,T_1 = \frac{1}{p}, \quad \text{Var } T_1 = \frac{1-p}{p^2},$$

$$M_{T_1}(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p).$$

There is no quick way to "see" the mean, variance and mgf. We will obtain the mgf from the pmf, and then use the mgf to get the mean and variance.

The mgf

Let $X \sim \text{Geometric}(p)$.

$$M_X(t) = E e^{tX} = \sum_{x \in \mathcal{X}} e^{tx} f_X(x)$$

$$= \sum_{x=1}^{\infty} \underbrace{e^{tx}}_{} (1-p)^{x-1} p$$

$$\hookrightarrow (e^t)^x = e^t \cdot (e^t)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} \left[(1-p)e^t\right]^{x-1}$$

A geometric series which converges if $(1-p)e^t < 1$, or equivalently $t < -\log(1-p)$

$$= \frac{pe^t}{1-(1-p)e^t} \quad \text{for } t < -\log(1-p)$$

(undefined otherwise)

Calculus Note:

$$\left(\frac{1}{V}\right)' = \frac{-V'}{V^2}$$

$$\left(\frac{1}{V}\right)'' = \frac{-V''}{V^2} + \frac{2(V')^2}{V^3}$$

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t} = p\left(\frac{1}{e^{-t}-(1-p)}\right)$$

$$= p \cdot \frac{1}{V}$$

$$M_X'(t) = \frac{+e^{-t} \cdot p}{(e^{-t}-(1-p))^2} = \frac{pe^t}{(1-(1-p)e^t)^2}$$

$$M_X''(t) = \frac{-e^{-t} \cdot p}{(e^{-t}-(1-p))^2} + \frac{2e^{-2t} \cdot p}{(e^{-t}-(1-p))^3}$$

$$= \frac{-pe^t}{(1-(1-p)e^t)^2} + \frac{2pe^t}{(1-(1-p)e^t)^3}$$

$$M'(0) = \frac{p}{p^2} = \boxed{\frac{1}{p} = EX}$$

$$M''(0) = \frac{-p}{p^2} + \frac{2p}{p^3} = -\frac{1}{p} + \frac{2}{p^2}.$$

$$\operatorname{Var} X = \left(-\frac{1}{p} + \frac{2}{p^2}\right) - \left(\frac{1}{p}\right)^2$$

$$= \frac{1}{p^2} - \frac{1}{p} = \boxed{\frac{1-p}{p^2}} = \operatorname{Var} X$$

The Discrete Memoryless Property

① If $X \sim \text{Geometric}(p)$, then X satisfies the Discrete Memoryless Property (DMP):

$$P(X > y+z \mid X > y) = P(X > z)$$
for all integers $y, z > 0$,

or equivalently,

$$P(X > y+z) = P(X > y)\,P(X > z) \qquad (\dagger)$$
for all integers $y, z > 0$.

Conversely,

② If the rv X has range $\mathcal{X} = \{1, 2, 3, \dots\}$ and X satisfies the DMP, then $X \sim \text{Geometric}(p)$ for some value of p.

Proof of ①:

$$P(X > y+z \mid X > y) = \frac{P(\{X > y+z\} \cap \{X > y\})}{P(X > y)}$$

$$= \frac{P(X > y+z)}{P(X > y)} \qquad (*)$$

Here we have used the fact that

$$\{X > y+z\} \subset \{X > y\}$$

so that $\{X > y+z\} \cap \{X > y\} = \{X > y+z\}$.

[In general: If event A implies event B, then $A \subset B$ so that $A \cap B = A$.]

Now note that $P(X > x) = (1-p)^x$, $x = 1, 2, 3, \ldots$

Plugging this into (*) yields

$$\frac{P(X > y+z)}{P(X > y)} = \frac{(1-p)^{y+z}}{(1-p)^y} = (1-p)^z$$

$$= P(X > z).$$

Proof of ②: Version (†) of DMP with

$z = 1$ gives

$$P(X > y+1) = P(X > y)\,P(X > 1) \text{ for } y = 1, 2, 3, \ldots$$

Setting $y = 1$ gives

$$P(X > 2) = P(X > 1)^2.$$

Then $y = 2$ gives

$$P(X > 3) = P(X > 2)\,P(X > 1) = P(X > 1)^3$$

The Negative Binomial Distribution

Let $X_1, X_2, X_3, \ldots$ be iid Bernoulli(p).

If T_r = the time of the r^{th} success
= the number of trials needed to obtain r successes

$$= \inf\{n : S_n = r\}$$

where $S_n = \sum_{i=1}^{n} X_i$,

then $T_r \sim$ Negative Binomial (r, p)

with pmf $P(T_r = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$

for $x = r, r+1, r+2, \ldots$

$$E\,T_r = \frac{r}{p}$$

$$\mathrm{Var}\,T_r = r\left(\frac{1-p}{p^2}\right)$$

mgf $M_{T_r}(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$ for $t < -\log(1-p)$

Argument:

$$T_r = T_1 + (T_2 - T_1) + \cdots + (T_r - T_{r-1})$$

Since the coin has no memory, it is intuitively obvious that

$T_1, T_2 - T_1, T_3 - T_2, \ldots, T_r - T_{r-1}$ are iid Geometric(p).

Therefore

$$ET_r = r(ET_1) = \frac{r}{p}$$

$$\operatorname{Var} T_r = r(\operatorname{Var} T_1) = r\left(\frac{1-p}{p^2}\right)$$

$$M_{T_r}(t) = \left(M_{T_1}(t)\right)^r = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$$

for $t < -\log(1-p)$.

Derivation of pmf:

$\{T_r = k\} = \{$ the k^{th} trial is a success and exactly $r-1$ of the previous trials are successes$\}$

$$\{T_r = k\} = \{X_k = 1\} \cap \{S_{k-1} = r-1\}$$

where S_n = # of successes in n trials,

and X_n = result of n^{th} trial.

Clearly $\{X_k = 1\}$ and $\{S_{k-1} = r-1\}$ are independent and $S_{k-1} \sim$ Binomial $(k-1, p)$.

Thus (for $k \geq r$)

$$P(T_r = k) = P(X_k = 1)\, P(S_{k-1} = r-1)$$

$$= p \binom{k-1}{r-1} p^{r-1} (1-p)^{k-r}$$

$$= \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Note: *My definition of the negative binomial is the book's "alternate definition".*

Given a sequence of independent coin tosses, each with probability p of heads, define

S_n = # of heads in n tosses,

T_r = # of tosses needed to get r heads.

We know that

$S_n \sim \text{Binomial}(n, p)$,

$T_r \sim \text{Neg. Bin}(r, p)$.

Finding the cdf of T_r

① Direct derivation (from defn. of cdf).

Recall $P(T_r = j) = \binom{j-1}{r-1} p^r (1-p)^{j-r}$

for $j = r, r+1, r+2, \ldots$

Thus $$P(T_r \le k) = \sum_{j=r}^{k} \binom{j-1}{r-1} p^r (1-p)^{j-r} \quad (*)$$

for integers $k \ge r$, and

$P(T_r \le k) = 0$ for $k < r$.

② Indirect derivation

Use the property

$$\{T_r > n\} = \{S_n < r\}.$$

This holds for <u>all</u> positive integers r and n.

Thus, for any positive integer k,

$$P(T_r \le k) = 1 - P(T_r > k) = 1 - P(S_k < r).$$

Now

$$P(S_k < r) = 1 \quad \text{if } k < r, \text{ and}$$

$$= \sum_{j=0}^{r-1} \binom{k}{j} p^j (1-p)^{k-j} \quad \text{if } k \ge r.$$

Thus

$$P(T_r \le k) = 1 - \sum_{j=0}^{r-1} \binom{k}{j} p^j (1-p)^{k-j} \quad \text{for } k \ge r$$

$$= 0 \quad \text{for } k < r.$$

This result and the earlier formula (*) give different formulas for the same quantity. For small r, this second formula is usually easier to use.

"Closure" Properties

Suppose X_1, X_2 are independent.

① If $X_1 \sim \text{Binomial}(n_1, p)$ and
$X_2 \sim \text{Binomial}(n_2, p)$,
then $X_1 + X_2 \sim \text{Binomial}(n_1 + n_2, p)$.

② If $X_1 \sim \text{Neg. Bin.}(r_1, p)$ and
$X_2 \sim \text{Neg. Bin.}(r_2, p)$,
then $X_1 + X_2 \sim \text{Neg. Bin.}(r_1 + r_2, p)$.

[Recall: Geometric(p) same as $NB(1, p)$.]

Similar properties hold for sums of three or more rv's.

These properties are intuitive. Give a "coin tossing" story for each.

Proofs: Use mgf's. Recall that mgf's are unique, and that

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$$

if X_1 and X_2 are independent.

① $M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$

$= (1-p+pe^t)^{n_1}(1-p+pe^t)^{n_2}$

$= (1-p+pe^t)^{n_1+n_2}$

$=$ mgf of Binomial(n_1+n_2, p).

Thus (by uniqueness)

$X_1+X_2 \sim$ Binomial(n_1+n_2, p).

② $M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$

$= \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^{r_1}\left(\dfrac{pe^t}{1-(1-p)e^t}\right)^{r_2}$

$= \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^{r_1+r_2}$

$=$ mgf of Neg.Bin(r_1+r_2, p).

Thus (by uniqueness of mgf's)

$X_1+X_2 \sim$ Neg.Bin(r_1+r_2, p).

Hypergeometric Distribution

(arises when sampling from a finite population with**out** replacement)

Suppose you have an urn with:

R red balls,
G green balls.

Draw k balls at random (with<u>out</u> replacement).

Define S_k = # of red balls in sample.

The pmf of S_k is

$$P(S_k = x) = \frac{\left(\begin{array}{c}\text{\# of samples of size } k \\ \text{containing exactly } x \\ \text{red balls}\end{array}\right)}{(\text{\# of samples of size } k)}$$

$$= \frac{\binom{R}{x}\binom{G}{k-x}}{\binom{R+G}{k}} \quad \text{for } 0 \le x \le R \text{ and } 0 \le k-x \le G.$$

(pmf $= 0$ otherwise.)

S_K has a hypergeometric distribution.

In the textbooks notation:

$S_K \sim \text{Hypergeometric}(N = R+G, M = R, K = k)$

<u>The mgf of S_K</u>

$S_K \in \{0, 1, 2, \ldots, k\}$

Since S_K is bounded, the mgf is finite for all t.

<u>But</u> the mgf has no simple closed form. (There is no way to simplify the summation.)

So mgf is not useful.

<u>Mean and Variance of S_K</u>

The book computes these directly from the pmf. We use another route.

Define $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ ball is red,} \\ 0 & \text{otherwise} \end{cases}$

$$= I_{\{i^{th} \text{ ball is red}\}}.$$

Then $S_K = \sum_{i=1}^{K} X_i$.

Since $EX_i = P(i^{th} \text{ ball is red})$

$$= P(1^{st} \text{ ball is red}) = \frac{R}{R+G},$$

(see discussion below)

we get

$$ES_K = E\sum_{i=1}^{K} X_i = \sum_{i=1}^{K} EX_i = K\frac{R}{R+G}.$$

<u>Discussion</u>: We are now thinking of the sample of K balls (chosen from R+G) as being ordered.

There are $(R+G)(R+G-1)\cdots(R+G-k+1)$

$$= \binom{R+G}{K} K! \text{ equally likely ordered samples.}$$

The number of ordered samples where the i^{th} ball is red is

$$R \cdot \binom{R+G-1}{K-1} \cdot (K-1)!$$

R: pick one red ball for the i^{th} position

$\binom{R+G-1}{K-1}$: pick the other $K-1$ balls

$(K-1)!$: Place them in the remaining $K-1$ positions

Thus

$$P(i^{th} \text{ ball is red}) = \frac{R \cdot \binom{R+G-1}{K-1} \cdot (K-1)!}{\binom{R+G}{K} K!}$$

$$= \frac{R \frac{(R+G-1)!}{(K-1)!(R+G-K)!} (K-1)!}{\frac{(R+G)!}{K!(R+G-K)!} K!} = \frac{R(R+G-1)!}{(R+G)!}$$

$$= \frac{R}{R+G}.$$

We will shortly need

For $i \neq j$,

$P(i^{th}$ and j^{th} balls are red$)$

$= P(1^{st}$ and 2^{nd} balls are red$)$

$$= \frac{R}{R+G} \cdot \frac{R-1}{R+G-1}.$$

A counting proof:

The number of ordered samples where the i^{th} <u>and</u> j^{th} balls are red is

$$\underset{\substack{\uparrow\\ \text{pick one red ball}\\ \text{for the } i^{th} \text{ position}}}{R} \cdot \underset{\substack{\uparrow\\ \text{pick one red ball}\\ \text{for the } j^{th} \text{ position}}}{(R-1)} \cdot \underset{\substack{\uparrow\\ \text{pick the other}\\ K-2 \text{ balls}}}{\binom{R+G-2}{K-2}} \cdot \underset{\substack{\uparrow\\ \text{Place them in the}\\ \text{remaining } K-2\\ \text{positions}}}{(K-2)!}$$

so that

$P(i^{th}$ and j^{th} balls are red$)$

$$= \frac{R(R-1)\binom{R+G-2}{K-2}(K-2)!}{\binom{R+G}{K}K!}$$

$$= \frac{R(R-1)(R+G-2)!}{(R+G)!} = \frac{R\cdot(R-1)}{(R+G)\cdot(R+G-1)}.$$

The Second Moment and Variance of the Hypergeometric Distribution

We want $\mathrm{Var}(S_K) = E\, S_K^2 - (E\, S_K)^2$.

We already know $E\, S_K = K \dfrac{R}{R+G}$.

$$S_K = \sum_{i=1}^{K} X_i \quad \text{so that}$$

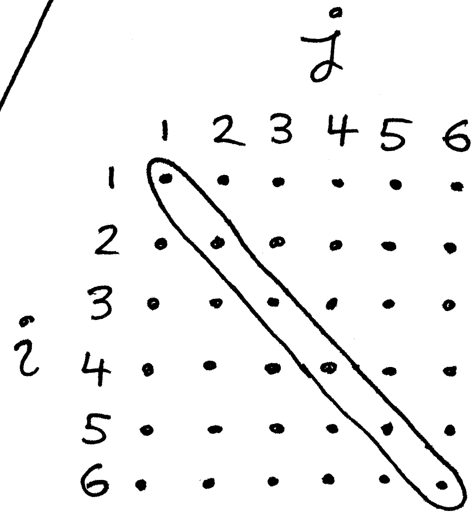
$$S_K^2 = \left(\sum_{i=1}^{K} X_i\right)^2 = \left(\sum_{i=1}^{K} X_i\right)\left(\sum_{j=1}^{K} X_j\right)$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} X_i X_j$$

and $$E\, S_K^2 = \underbrace{\sum_{i=1}^{K} \sum_{j=1}^{K}}_{} E\, X_i X_j$$

K^2 terms in which K terms have $i = j$ and $K^2 - K = K(K-1)$ have $i \neq j$

Case of $K=6$



Note that

$$X_i X_j = I_{\{i^{\text{th}}\text{ ball red}\}} I_{\{j^{\text{th}}\text{ ball red}\}}$$

$$= I_{\{i^{\text{th}}\text{ and } j^{\text{th}}\text{ ball are red}\}}$$

(recall $I_A I_B = I_{A \cap B}$.)

so that

$$E X_i X_j = P(i^{\text{th}} \text{ and } j^{\text{th}} \text{ balls are red})$$

(recall $E I_C = P(C)$)

$$= \begin{cases} \frac{R}{R+G} \cdot \frac{R-1}{R+G-1} & \text{for } i \neq j \\ \frac{R}{R+G} & \text{for } i = j. \end{cases}$$

Thus $ES_K^2 = \sum_{i=1}^{K} \sum_{j=1}^{K} E X_i X_j$

$$= K \frac{R}{R+G} + K(K-1) \frac{R}{R+G} \cdot \frac{R-1}{R+G-1}.$$

$$\mathrm{Var}(S_K) = E S_K^2 - (E S_K)^2$$

$$= K \frac{R}{R+G} + K(K-1) \frac{R}{R+G} \cdot \frac{R-1}{R+G-1} - \left(K \frac{R}{R+G}\right)^2$$

$$= K \frac{R}{R+G} \frac{\left[(R+G)(R+G-1) + (K-1)(R-1)(R+G) - K R (R+G-1) \right]}{(R+G)(R+G-1)}$$

$$= \underbrace{K \frac{R}{R+G} \cdot \frac{G}{R+G}}_{\text{Variance of Binomial}\left(n=K,\, p=\frac{R}{R+G}\right)} \cdot \underbrace{\frac{R+G-K}{R+G-1}}_{\text{finite population correction factor}}$$

The Poisson Distribution

If $X \sim \text{Poisson}(\lambda)$, then

pmf $P(X=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$

$EX = \text{Var}\, X = \lambda$

mgf $M_X(t) = e^{\lambda(e^t - 1)}$ (exercise)

Closure Property

If $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$, and X_1 and X_2 are independent,

then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Proof:

$$M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t) \quad \text{by independence}$$

$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$$= \text{mgf of Poisson}(\lambda_1+\lambda_2)$$

The Poisson distn. arises because of

Poisson approximation to the Binomial distn.

If $X \sim \text{Binomial}(n, p)$ where n is large and p is small,

then $P(X=k) \approx \dfrac{\lambda^k e^{-\lambda}}{k!}$ with $\lambda = np$

(that is, $X \sim$ approx Poisson$(\lambda = np)$).

Formal statement as a limit theorem

Fix a value $\lambda > 0$.

Suppose $X_n \sim \text{Binomial}(n, p = \frac{\lambda}{n})$ for $n = 1, 2, 3, \ldots$

and $Y \sim \text{Poisson}(\lambda)$. Then

$$\lim_{n \to \infty} P(X_n = k) = \frac{\lambda^k e^{-\lambda}}{k!} = P(Y = k),$$

that is, $X_n \xrightarrow{d} Y$ (convergence in distn.)

Poisson approximation to hypergeometric distn.

If $X \sim \text{hypergeometric}(N, M, k)$ and

[# of red balls in sample of k] [total # of balls] [# of red balls in urn]

N is large, $\frac{M}{N}$ is small, and $\frac{k}{N}$ is small,

then $X \sim$ approx Poisson$(\lambda = k \cdot \frac{M}{N})$.

More general Poisson approximation

Suppose $A_1, A_2, \ldots, A_n$ are independent events and $Z_1, Z_2, \ldots, Z_n$ are the corresponding indicator random variables

$$Z_i = I_{A_i} = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Let $p_i = P(A_i)$.

If n is large and all the p_i are small, then

$$S = \sum_{i=1}^{n} Z_i$$

has approximately a $\text{Poisson}\left(\lambda = \overbrace{\sum_{i=1}^{n} p_i}^{= ES}\right)$ distn.

Examples

1. Geiger counter with lump of radioactive material consisting of different isotopes.
2. Traffic accidents (on a certain highway during a given period) with drivers of differing ability.
3. Number of cases of a rare disease (in a given city) when people vary in their susceptibility.

etc.

Example: Suppose

2000 skiers at a resort,

each has probability .002 of an accident on any given day,

skiers are independent.

Then

$$X = \#\text{ of accidents today}$$

$$\sim \text{Binomial}(\underset{(\text{large})}{n = 2000}, \underset{(\text{small})}{p = .002})$$

$$\sim \text{approx Poisson}(\lambda = 2000(.002) = 4).$$

Thus $P(X=2) \approx \dfrac{\lambda^2 e^{-\lambda}}{2!} = \dfrac{4^2 e^{-4}}{2}$.

Continuation: Similar situation but now

$$P(\text{skier } i \text{ has accident}) = p_i = 2 \times 10^{-6} i$$

$$\text{for } i = 1, 2, \ldots, 2000.$$

Maximum $p_i = p_{2000} = 2 \times 10^{-6} \times 2000$

$= 4 \times 10^{-3}$ which is "small".

Thus $X \sim \text{approx Poisson}(\lambda = \sum_{i=1}^{2000} p_i)$.

$$\lambda = 2 \times 10^{-6} \sum_{i=1}^{2000} i = 2 \times 10^{-6} \cdot \frac{2000 \cdot 2001}{2} \approx 4$$

So $P(X=2) \approx$ same as before.

The Poisson Process

A process of arrivals (clicks, events, etc.) is called a <u>Poisson process with constant rate λ</u> if

1. The number of arrivals in any given fixed period of time of duration t has a Poisson(λt) distribution.
2. Disjoint intervals of time are independent.

mean

Consequence of 1 : The expected number of arrivals in a period of length t is λt. The arrivals occur at rate λ (on average).

<u>Examples</u>:

1. Clicks on a Geiger counter (under uniform conditions)
2. Accidents on a highway (under uniform conditions)

etc.

Example: Clicks on a Geiger counter occur according to a Poisson process with an average rate of 1.5 clicks per second.

What is the probability there are exactly 2 clicks during the time interval $(2.5, 5.5)$ and exactly 3 clicks during $(5.5, 9.5)$?

Answer: Define

$$\begin{aligned} X_1 &= \# \text{ of clicks during } (2.5, 5.5), \\ X_2 &= \# \text{ of clicks during } (5.5, 9.5). \end{aligned}$$

Then

$$\begin{aligned} X_1 &\sim \text{Poisson}\,(\lambda_1 = 1.5 \times (5.5 - 2.5) = 1.5 \times 3 = 4.5) \\ X_2 &\sim \text{Poisson}\,(\lambda_2 = 1.5 \times (9.9 - 5.5) = 1.5 \times 4 = 6.0)\ , \end{aligned}$$

and X_1 and X_2 are **independent because the time intervals $(2.5, 5.5)$ and $(5.5, 9.5)$ are disjoint.**

$$\begin{aligned} P(X_1 = 2, X_2 = 3) &= P(X_1 = 2)P(X_2 = 3) \quad \text{by independence} \\ &= \frac{\lambda_1^2 e^{-\lambda_1}}{2!} \cdot \frac{\lambda_2^3 e^{-\lambda_2}}{3!} \\ &= \frac{4.5^2 e^{-4.5}}{2!} \cdot \frac{6^3 e^{-6}}{3!} \\ &= 0.1124786 \times 0.08923508 \\ &= 0.01003704 \end{aligned}$$

Example (Poisson Process)

Clicks on a Geiger counter occur according to a Poisson process with average rate 1.5 clicks per second.

What is the probability of

exactly 2 clicks during $(0,3)$

and exactly 3 clicks during $(2,6)$?

These time periods overlap.

So we do the following.

Let
$$\left.\begin{aligned} X_1 &= \#\text{ of clicks during } (0,2), \\ X_2 &= \#\text{ of clicks during } (2,3), \\ X_3 &= \#\text{ of clicks during } (3,6). \end{aligned}\right\} \text{disjoint intervals of time}$$

Then X_1, X_2, X_3 are independent with

$$X_1 \sim \text{Poisson}(\lambda_1 = 1.5 \times 2 = 3)$$
$$X_2 \sim \text{Poisson}(\lambda_2 = 1.5 \times 1 = 1.5)$$
$$X_3 \sim \text{Poisson}(\lambda_3 = 1.5 \times 3 = 4.5).$$

We want $P(X_1 + X_2 = 2, X_2 + X_3 = 3)$

$$= P(X_1 = 2, X_2 = 0, X_3 = 3)$$
$$+ P(X_1 = 1, X_2 = 1, X_3 = 2)$$
$$+ P(X_1 = 0, X_2 = 2, X_3 = 1)$$

$$= P(X_1=2)P(X_2=0)P(X_3=3)$$
$$+ P(X_1=1)P(X_2=1)P(X_3=2)$$
$$+ P(X_1=0)P(X_2=2)P(X_3=1)$$

$$= \frac{3^2 e^{-3}}{2!} \times \frac{1.5^0 e^{-1.5}}{0!} \times \frac{4.5^3 e^{-4.5}}{3!}$$
$$+ \frac{3^1 e^{-3}}{1!} \times \frac{1.5^1 e^{-1.5}}{1!} \times \frac{4.5^2 e^{-4.5}}{2!}$$
$$+ \frac{3^0 e^{-3}}{0!} \times \frac{1.5^2 e^{-1.5}}{2!} \times \frac{4.5^1 e^{-4.5}}{1!}$$

$$= \left(\frac{9 \cdot 4.5^3}{12} + \frac{3 \cdot 1.5 \cdot 4.5^2}{2} + \frac{1.5^2 \cdot 4.5}{2}\right) e^{-9}$$