Continuous Distributions

The Gamma dist.

$X \sim \text{Gamma}(\alpha, \beta)$ has

pdf $f(x) = \dfrac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$ for $x > 0$.
$(\alpha > 0, \beta > 0)$

$EX = \alpha\beta$

$\text{Var}X = \alpha\beta^2$

mgf $M_X(t) = \left(\dfrac{1}{1-\beta t}\right)^\alpha$, $t < \frac{1}{\beta}$.

Calculation of moments done earlier.

Derivation of mgf:

$$E e^{tX} = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_0^\infty e^{tx} \cdot \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}\,dx$$

$$= \frac{1}{\beta^\alpha} \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x\left(\frac{1}{\beta}-t\right)}\,dx$$

which is finite when $t < \frac{1}{\beta}$

Now let $\frac{1}{\beta'} = \frac{1}{\beta} - t$

so that $\beta' = \dfrac{1}{\frac{1}{\beta} - t} = \dfrac{\beta}{1-\beta t}$.

Then integral becomes

$$= \frac{(\beta')^\alpha}{\beta^\alpha} \int_0^\infty \underbrace{\frac{x^{\alpha-1} e^{-x/\beta'}}{\Gamma(\alpha)(\beta')^\alpha}}_{\text{Gamma}(\alpha,\beta')\text{ pdf}} dx$$

$$= \left(\frac{\beta'}{\beta}\right)^\alpha \cdot 1 = \left(\frac{1}{1-\beta t}\right)^\alpha \text{ for } t < \frac{1}{\beta}.$$

<u>Closure Property</u>

If X_1, X_2 independent

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$,

$X_2 \sim \text{Gamma}(\alpha_2, \beta)$,

then $X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$

<u>Proof</u>: $M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \left(\frac{1}{1-\beta t}\right)^{\alpha_2} = \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2}$$

$=$ mgf of $\text{Gamma}(\alpha_1 + \alpha_2, \beta)$. QED.

The Exponential Distribution ($\text{Exp}(\beta)$)

pdf $f_X(x) = \frac{1}{\beta} e^{-x/\beta}$ for $x > 0$ $(\beta > 0)$,

$EX = \beta$, $\text{Var } X = \beta^2$,

cdf $F_X(x) = 1 - e^{-x/\beta}$ for $x > 0$,

$$P(X > x) = e^{-x/\beta} \text{ for } x > 0,$$

mgf $M_X(t) = \frac{1}{1-\beta t}$ for $t < 1/\beta$,

$\text{Exp}(\beta)$ same as $\text{Gamma}(1, \beta)$.

The (Continuous) Memoryless Property

If $X \sim \text{Exp}(\beta)$, then

$$P(X - t > s \mid X > t) = P(X > s) \qquad (*)$$

for <u>all</u> $s, t > 0$

or equivalently,

$$P(X > s + t) = P(X > s) P(X > t)$$

for <u>all</u> $s, t > 0$

<u>Converse</u> : If the r.v. Y satisfies $(*)$, then $Y \sim \text{Exp}(\beta)$ for some value of β.

Failure rate (hazard rate)

Suppose X has pdf $f(x)$, cdf $F(x)$,
and $f(x) = 0$ for $x < 0$.

Define the hazard function

$$h(t) = \lim_{\delta \downarrow 0} \frac{P(t < X \leq t+\delta \mid X > t)}{\delta}$$

$$\left(= \frac{f(t)}{1 - F(t)} \text{ by exercise 3.25} \right).$$

Then $P(t < X \leq t+\delta \mid X > t) \approx h(t)\delta$
for small δ.

Fact: $X \sim \exp(\beta)$ iff $h(t) = \frac{1}{\beta}$
for all t.

The exponential distn. has a constant hazard rate, and it is the only distn. with this property.

(Note: "constant hazard rate" property is equivalent to "memoryless" property.)

Proof of one direction:

Suppose $X \sim \exp(\beta)$. Then

$$h(t) = \lim_{\delta \downarrow 0} \frac{P(t < X \leq t+\delta \mid X > t)}{\delta}$$

$$= \lim_{\delta \downarrow 0} \frac{1}{\delta} \frac{P(t < X \leq t+\delta)}{P(X > t)}$$

$$= \lim_{\delta \downarrow 0} \frac{1}{\delta} \frac{F(t+\delta) - F(t)}{1 - F(t)}$$

$$= \lim_{\delta \downarrow 0} \frac{(1 - e^{-(t+\delta)/\beta}) - (1 - e^{-t/\beta})}{\delta e^{-t/\beta}}$$

$$= \lim_{\delta \downarrow 0} \left(\frac{1 - e^{-\delta/\beta}}{\delta} \right) = \frac{1}{\beta}$$

by L'Hospital's.

Fact: If $Z_1, Z_2, \ldots, Z_n$ are iid $\exp(\beta)$, then $X \equiv \min Z_i \sim \exp(\beta/n)$.

"**Proof**": (heuristic)

X has the memoryless property.

Thus $X \sim \exp(\xi)$ for some value ξ.

What is ξ?

Determine ξ by considering the failure rate.

"Clearly"

$$h_X(t) = n h_{Z_1}(t) = n \cdot \frac{1}{\beta}.$$

But $h_X(t) = \frac{1}{\xi}$.

Thus $\frac{1}{\xi} = \frac{n}{\beta} \Rightarrow \xi = \frac{\beta}{n}$.

A more formal proof uses:

Lemma: If $X_1, X_2, \ldots, X_n$ are independent with cdf's $F_1, F_2, \ldots, F_n$, then

$$F_{\min X_i}(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)) \quad \text{and}$$

$$F_{\max X_i}(t) = \prod_{i=1}^{n} F_i(t).$$

Consequences of Lemma:

If $X_1, \ldots, X_n$ are i.i.d. with cdf F, then

$$F_{\min X_i}(t) = 1 - (1 - F(t))^n$$

$$F_{\max X_i}(t) = (F(t))^n$$

If $X_1, \ldots, X_n$ are iid $\exp(\beta)$ rv's, then

$$F_{\min X_i}(t) = 1 - (e^{-t/\beta})^n = 1 - e^{-t/(\beta/n)} \quad \text{for } t \geq 0,$$

(Thus $\min X_i \sim \exp(\beta/n)$.)

$$F_{\max X_i}(t) = (1 - e^{-t/\beta})^n \quad \text{for } t \geq 0.$$

Proof of Lemma:

$$F_{\max X_i}(t) = P\left(\max_{1\leq i\leq n} X_i \leq t\right)$$

$$= P\left(\bigcap_{i=1}^{n} \{X_i \leq t\}\right)$$

(since $\max X_i \leq t$ if and only if all the values X_i are $\leq t$)

$$= \prod_{i=1}^{n} P(X_i \leq t) \quad \text{by independence}$$

$$= \prod_{i=1}^{n} F_i(t)$$

$$F_{\min X_i}(t) = P(\min X_i \leq t) = 1 - P(\min X_i > t)$$

$$= 1 - P\left(\bigcap_{i=1}^{n} \{X_i > t\}\right)$$

(since $\min X_i > t$ iff all the values X_i are $> t$)

$$= 1 - \prod_{i=1}^{n} P(X_i > t) = 1 - \prod_{i=1}^{n}(1 - F_i(t))$$

Problem: (The "K stall" problem.)

A room is lit by K light bulbs. The lifetimes of the bulbs are iid Exp(β). How long until the room is in total darkness?

Find the mean and variance of the time until total darkness.

Solution: Assume K lightbulbs with lifetimes $Z_1, Z_2, \ldots, Z_K \sim \text{iid } \exp(\beta)$.

Define

$$Z_{(1)} = Z_{(1:K)} = \min(Z_1, Z_2, \ldots, Z_K)$$
= time at which first bulb burns out,

$$Z_{(i)} = Z_{(i:K)}$$ = time at which i^{th} bulb burns out,

$$Z_{(K)} = Z_{(K:K)}$$ = time at which last bulb burns out

= time until total darkness

$$= \max(Z_1, Z_2, \ldots, Z_K).$$

Note that $Z_{(1)} < Z_{(2)} < \cdots < Z_{(K)}$ with probability one (that is, ties have probability zero).

Consider the identity

$$Z_{(K)} = Z_{(1)} + (Z_{(2)} - Z_{(1)}) + \cdots + (Z_{(K)} - Z_{(K-1)}).$$

The Memoryless Property implies:

If at any time j bulbs remain, the amount of time until the <u>next</u> bulb burns out does not depend on how long the bulbs have been in operation.

At time $Z_{(i:K)}$ there are $K-i$ bulbs remaining. The memoryless property says they are all as good as new. The time until the next bulb burns out is the minimum of $K-i$ exponential rv's which are iid $\exp(\beta)$. Thus

$$Z_{(i+1:K)} - Z_{(i:K)} \sim \exp\left(\frac{\beta}{K-i}\right)$$

and this rv is independent of what has happened already $Z_{(1)}, Z_{(2)}, \ldots, Z_{(i)}$.

Thus (intuitively)

$$Z_{(1)}, Z_{(2)} - Z_{(1)}, \ldots, Z_{(K)} - Z_{(K-1)}$$

are <u>independent</u> exponential rv's

with $Z_{(1)} \sim \exp\left(\frac{\beta}{K}\right)$ and

$$Z_{(i+1)} - Z_{(i)} \sim \exp\left(\frac{\beta}{K-i}\right).$$

Thus

$$Z_{(K)} = Z_{(1)} + (Z_{(2)} - Z_{(1)}) + \cdots + (Z_{(K)} - Z_{(K-1)})$$

implies

$$\begin{aligned} E Z_{(K)} &= E Z_{(1)} + E(Z_{(2)} - Z_{(1)}) + \cdots \\ &\qquad + E(Z_{(K)} - Z_{(K-1)}) \\ &= \frac{\beta}{K} + \frac{\beta}{K-1} + \frac{\beta}{K-2} + \cdots + \frac{\beta}{2} + \beta \end{aligned}$$

and (by independence)

$$\begin{aligned} \operatorname{Var} Z_{(K)} &= \operatorname{Var}(Z_{(1)}) + \operatorname{Var}(Z_{(2)} - Z_{(1)}) \\ &\qquad + \cdots + \operatorname{Var}(Z_{(K)} - Z_{(K-1)}) \\ &= \left(\frac{\beta}{K}\right)^2 + \left(\frac{\beta}{K-1}\right)^2 + \cdots + \left(\frac{\beta}{2}\right)^2 + \beta^2. \end{aligned}$$

Alternate Solution (Analytic):

$$F_X(t) = F_{\max Z_i}(t) = (F_{Z_1}(t))^k = (1 - e^{-t/\beta})^k$$

so that the pdf of X is

$$f_X(t) = F'_X(t) = k(1 - e^{-t/\beta})^{k-1}\left(\frac{1}{\beta}e^{-t/\beta}\right) \text{ for } t > 0\,.$$

Using the pdf we can compute the mean and variance of X in the usual way.

Another way to compute the mean is to use the result of a homework exercise:

$$\begin{aligned} EX &= \int_0^\infty (1 - F_X(t))\,dt \quad \text{for any nonnegative rv } X \\ &= \int_0^\infty (1 - (1 - e^{-t/\beta})^k)\,dt \quad \text{in this case.} \end{aligned}$$

Setting $k = 3$ (just for example) and expanding the power gives

$$\begin{aligned} EX &= \int_0^\infty (1 - (1 - e^{-t/\beta})^3)\,dt \\ &= \int_0^\infty (1 - (1 - 3e^{-t/\beta} + 3e^{-2t/\beta} - e^{-3t/\beta}))\,dt \\ &= 3\beta - 3\left(\frac{\beta}{2}\right) + \frac{\beta}{3} \quad \text{which agrees with} \\ &= \frac{\beta}{3} + \frac{\beta}{2} + \beta \quad \text{from the earlier approach.} \end{aligned}$$

We can also obtain the variance in a similar fashion.

<u>Fact</u>: For a Poisson process with rate λ, the time until the first arrival has an exponential distribution with mean

$$\beta = \frac{1}{\lambda}.$$

<u>Proof</u>: (Start time from zero.)

Let T_1 = the time of the first arrival,

S_t = the # of arrivals by time t.

We know $S_t \sim \text{Poisson}(\lambda t)$ so that

$$P(S_t = 0) = \left.\frac{(\lambda t)^k e^{-\lambda t}}{k!}\right|_{k=0} = e^{-\lambda t}.$$

But $\{T_1 > t\} = \{S_t = 0\}$ (Think about it!)

Thus $P(T_1 > t) = P(S_t = 0) = e^{-\lambda t}$

and $P(T_1 \le t) = 1 - e^{-\lambda t}$.

This is the cdf of $\text{Exp}(1/\lambda)$ distn.

A Connection between the Gamma distn. and the Poisson distn.

(via the Poisson process)

Fact: A process of arrivals is a Poison process with rate λ

if and only if

the interarrival times are i.i.d. exponential rv's with mean $\frac{1}{\lambda}$.

Notation: Let (time starts from zero)

T_r = time of r^{th} arrival,

S_t = # of arrivals during $(0, t)$.

The interarrival times are

$$T_1, T_2 - T_1, T_3 - T_2, \ldots, T_r - T_{r-1}, \ldots$$

Note that

$T_r \sim \text{Gamma}(r, \frac{1}{\lambda})$ (by fact above)

$S_t \sim \text{Poisson}(\lambda t)$ (by defn. of Poisson process)

Thus

$$\{T_r > t\} = \{S_t < r\}$$

implies $P(T_r > t) = P(S_t < r)$

which leads to

$$\int_t^\infty \underbrace{\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}}_{\substack{\text{Gamma}(\alpha = r,\, \beta = \frac{1}{\lambda}) \\ \text{pdf}}} dx = \sum_{i=0}^{r-1} \underbrace{\frac{(\lambda t)^i e^{-\lambda t}}{i!}}_{\substack{\text{Poisson}(\lambda t) \\ \text{pmf}}} .$$

This formula can also be derived by repeated integration by parts.

Law of Large Numbers (LLN)

Let $X_1, X_2, X_3, \ldots$ be iid with finite mean μ.
(That is, $E|X_i| < \infty$ and $EX_i = \mu$.)

Define $S_n = \sum_{i=1}^{n} X_i$ and $\bar{X}_n = \frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Strong Law of Large Numbers (SLLN):

$$P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$$

In words: the sequence $\bar{X}_n$, $n = 1, 2, 3, \ldots$, converges to μ with probability 1 (as $n \to \infty$).

This result implies the ...

Weak Law of Large Numbers (WLLN):

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1 \quad \text{for all } \varepsilon > 0\,,$$

$$\text{or} \qquad \lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0\,.$$

• If $X_1, X_2, X_3, \ldots$ also has a finite variance $\text{Var}(X_i) = \sigma^2 < \infty$, then we can say more:

$$\text{Var}(S_n) = n\sigma^2\,, \qquad \text{Var}(\bar{X}_n) = \text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\text{Var}(S_n) = \frac{\sigma^2}{n}\,,$$

and Chebyshev's Inequality (see text, p. 122) implies

$$P(|\bar{X}_n - \mu| > \varepsilon) < \frac{\sigma^2}{n\varepsilon^2} \quad (\text{which} \to 0 \text{ as } n \to \infty)\,.$$

Central Limit Theorem (CLT)

If $X_1, X_2, \ldots, X_n$ are iid from a distribution with $E X_i = \mu$ and $\operatorname{Var} X_i = \sigma^2 < \infty$,

then
$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} N(0,1) \quad (\text{as } n \to \infty)$$

and, equivalently,
$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad (\text{as } n \to \infty)$$

where $S_n = \sum_{i=1}^{n} X_i$ and $\bar{X}_n = S_n/n$.

Informal Statement

For "large" n,

$$S_n \sim \text{approx } N(n\mu, n\sigma^2),$$
$$\bar{X}_n \sim \text{approx } N\left(\mu, \frac{\sigma^2}{n}\right).$$

How large should n be?

Normal Distribution

If $X \sim N(\mu, \sigma^2)$, then

$$X + c \sim N(\mu + c, \sigma^2)$$

$$cX \sim N(c\mu, c^2\sigma^2)$$

$$aX + b \sim N(a\mu + b, a^2\sigma^2)$$

$$\frac{X - \mu}{\sigma} \sim N(0,1)$$

$$P(X \le b) = P\left(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right), \text{ etc.}$$

Normal Approximation

Suppose X has mean μ, variance σ^2, and a distribution which is approximately normal.

Let $X^* \sim N(\mu, \sigma^2)$.

$$P(X \le b) \approx P(X^* \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right)$$

The Continuity Correction

The normal distribution is continuous. If a normal approximation is used for a discrete distribution, caution is required.

When using a normal approximation to the distribution of an **integer-valued** random variable X, greater accuracy is usually obtained by using the "continuity correction".

Let X^* be a normal random variable with the same mean and variance as X. For integers b and c, the continuity correction is:

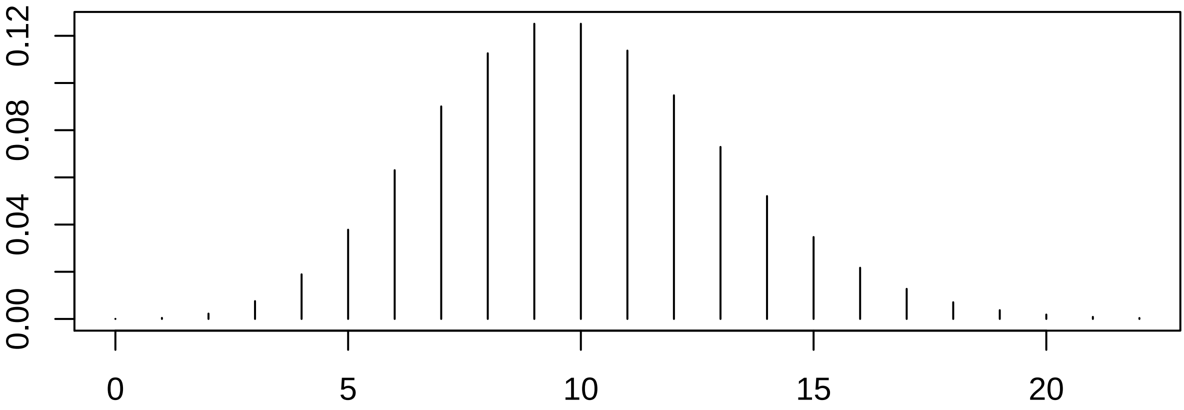
$$\begin{aligned} P(X = b) &\approx P(X^* \in [b - .5\,,\, b + .5]\,) \\ P(b \leq X \leq c) &\approx P(b - .5 < X^* < c + .5) \\ P(X \geq b) &\approx P(X^* > b - .5) \\ P(X \leq c) &\approx P(X^* < c + .5) \end{aligned}$$

The continuity correction essentially amounts to replacing each of the "spikes" in the pmf of X by a rectangle of the same height with width one. This converts the pmf (a series of spikes) into a density (pdf) which looks like a histogram (a series of rectangles). The normal distribution is then used as an approximation to this histogram.

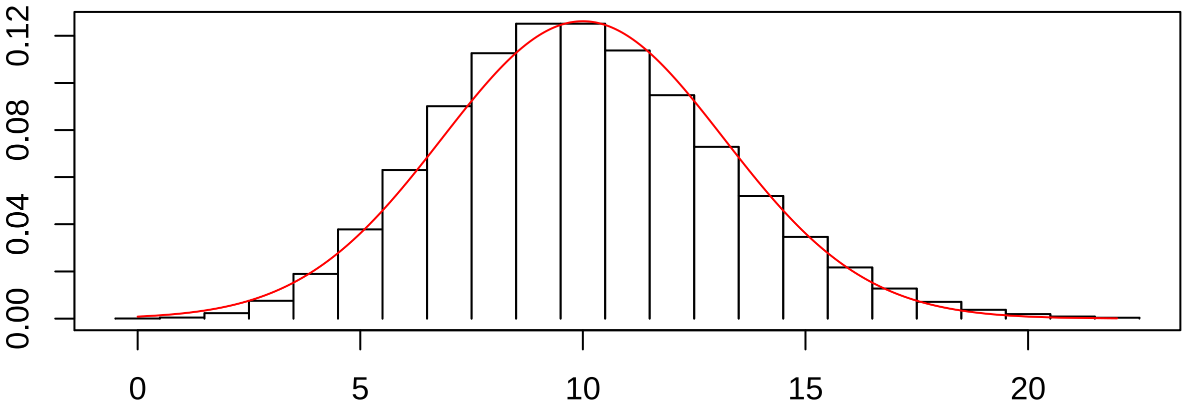
After using the continuity correction, we standardize the random variable X^* in the usual way. If X and X^* have mean μ and variance σ^2, then $Z = \dfrac{X^* - \mu}{\sigma} \sim N(0, 1)$. For example

$$P(b \leq X \leq c) \approx P\left(\frac{b - .5 - \mu}{\sigma} < Z < \frac{c + .5 - \mu}{\sigma}\right).$$

Poisson pmf (lambda=10)



Corresponding pdf + Normal(10,10) pdf



Normal Approximations

Binomial distn.

If $X \sim \text{Binomial}(n, p)$ and n is large, then

$$X \sim \text{approx Normal}(\mu = np, \sigma^2 = np(1-p)).$$

Argument: $X = \sum_{i=1}^{n} Z_i$ where

$Z_1, Z_2, \ldots, Z_n$ are iid Bernoulli(p).

Now use CLT.

Poisson distn.

If $X \sim \text{Poisson}(\lambda)$ and λ is large,

then $X \sim \text{approx Normal}(\mu = \lambda, \sigma^2 = \lambda)$.

Argument: Fix a value λ_0.

Let $Z_1, Z_2, \ldots, Z_n$ be iid Poisson(λ_0).

Then $\sum_{i=1}^{n} Z_i \sim \text{Poisson}(\lambda = n\lambda_0)$.

Now let $n \to \infty$ and use CLT.

Negative Binomial distn

If $X \sim \text{Neg Bin}(r, p)$ and r is large,
then $X \sim \text{approx } N\left(\mu = \frac{r}{p}, \sigma^2 = r \cdot \frac{1-p}{p^2}\right)$

Argument: $X = \sum_{i=1}^{r} Z_i$ where
$Z_1, Z_2, \ldots, Z_r$ are iid Geometric(p)
Now use CLT.

Gamma distn.

If $X \sim \text{Gamma}(\alpha, \beta)$ and α is large,
then X is approximately Normal.

Argument: Fix a value α_0.
Let $Z_1, Z_2, \ldots, Z_n$ be iid Gamma(α_0, β).
Then $\sum_{i=1}^{n} Z_i \sim \text{Gamma}(n\alpha_0, \beta)$.
Now let $n \to \infty$ and use CLT.

Beta distn

If $X \sim \text{Beta}(\alpha, \beta)$ with both α, β large,
then X is approx. Normal.

and others...

Comments on C5

Suppose

n people support candidate A.
n people support B.
Each person votes with probability p.
People decide independently.

Let X = # who vote for A,
Y = # who vote for B.

Then X and Y are independent with a Binomial (n, p) distribution which is approximately

Normal $(\mu = np, \sigma^2 = np(1-p))$

if n is sufficiently large.

$P(\text{tie}) = P(X = Y) = P(X - Y = 0)$.

Recall: If X, Y independent with

$X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$,

then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

and $X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.

Thus, for n large,

$$D \equiv X - Y \sim N(\mu - \mu, \sigma^2 + \sigma^2)$$ (approximately)

$$\sim N(0, 2\sigma^2)$$

Let $D^* \sim N(0, 2\sigma^2)$.

$\left(\begin{array}{l} D \text{ is approx } N(0, 2\sigma^2). \\ D^* \text{ is exactly } N(0, 2\sigma^2) \end{array} \right)$

Since D is integer-valued, the continuity correction gives

$$P(D=0) \approx P\left(-\tfrac{1}{2} < D^* < \tfrac{1}{2}\right)$$

$$= P\left(\frac{-\frac{1}{2} - 0}{\sqrt{2\sigma^2}} < \frac{D^* - 0}{\sqrt{2\sigma^2}} < \frac{\frac{1}{2} - 0}{\sqrt{2\sigma^2}} \right)$$

$$= P(-\varepsilon < Z < \varepsilon)$$

where $Z = \dfrac{D^* - 0}{\sqrt{2\sigma^2}} \sim N(0,1)$

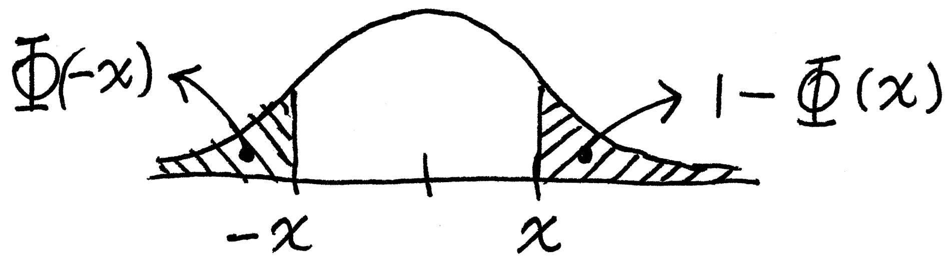
and $\varepsilon = (\frac{1}{2} - 0)/\sqrt{2\sigma^2}$.

Note: If n is large, $\sigma^2 = np(1-p)$ is also large so that ε will be small.

$= \Phi(\varepsilon) - \Phi(-\varepsilon)$ where Φ is the cdf of $N(0,1)$

$= \Phi(\varepsilon) - (1 - \Phi(\varepsilon)) = 2\Phi(\varepsilon) - 1$

by symmetry of the $N(0,1)$ distn.



$$\Phi(-x) = 1 - \Phi(x) \text{ for all } x.$$

Thus, to compute $P(\text{tie})$ you calculate $\varepsilon = \dfrac{1}{2\sqrt{2np(1-p)}}$

and use tables or computer to get

$$2\Phi(\varepsilon) - 1.$$

Problem with Tables

Tables typically give $\Phi(z)$ or $P(Z > z)$ for $z = 0.00, 0.01, 0.02, \ldots, 3.49$ (say).

If ε is small, rounding to the nearest tabled z-value loses a lot of accuracy.

Better to use ...

Good approximation for small ϵ

Let $\varphi(z) = \dfrac{e^{-z^2/2}}{\sqrt{2\pi}}$ = pdf for $N(0,1)$.

Then $P(\text{tie})$ is approximately

$$P(-\epsilon < \underset{\substack{\uparrow \\ N(0,1)}}{Z} < \epsilon) = \int_{-\epsilon}^{\epsilon} \varphi(z)\,dz$$

$$\approx (2\epsilon)\,\varphi(0) = \frac{2\epsilon}{\sqrt{2\pi}}$$ when ϵ is small.

Thus $P(\text{tie}) \approx \dfrac{2}{\sqrt{2\pi}} \cdot \dfrac{1}{2\sqrt{2np(1-p)}}$

$$= \frac{1}{\sqrt{4\pi n p(1-p)}}.$$

When $n = 2{,}000{,}000$, and $p = 0.8$ you get

$P(\text{tie}) \approx .0004987$

Note that $\epsilon = .000625$ which rounds to zero giving $2\underbrace{\Phi(0)}_{.5} - 1 = 0$.

Problem actually wants

$P(\text{Joe makes a difference})$

$= P(\text{Joe breaks a tie}) + P(\text{Joe creates a tie})$

$= P(D=0) + P(D=1)$

(assuming Joe votes for B)

$= P(0 \leq D \leq 1)$

$\approx P(-.5 \leq D^* \leq 1.5)$

$$= P\left(\frac{-.5}{\sqrt{2\sigma^2}} < Z < \frac{1.5}{\sqrt{2\sigma^2}}\right)$$

$$\approx \underbrace{\left[\frac{1.5}{\sqrt{2\sigma^2}} - \left(\frac{-.5}{\sqrt{2\sigma^2}}\right)\right]}_{\text{width of interval of integration}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}}}_{\text{density of } Z \text{ at zero}} \quad \left(\text{when } n \text{ is large}\right)$$

$$= \frac{2}{\sqrt{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{\pi\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi n p(1-p)}} = .000997$$

when $n = 2000000$, $p = .8$.

Beta Distribution

$X \sim \text{Beta}(\alpha, \beta)$ with $\alpha > 0, \beta > 0$
has pdf

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for } 0<x<1$$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Parameters α, β are "shape" parameters.

$$\alpha<1 \Rightarrow \lim_{x\to 0^+} f(x) = \infty$$

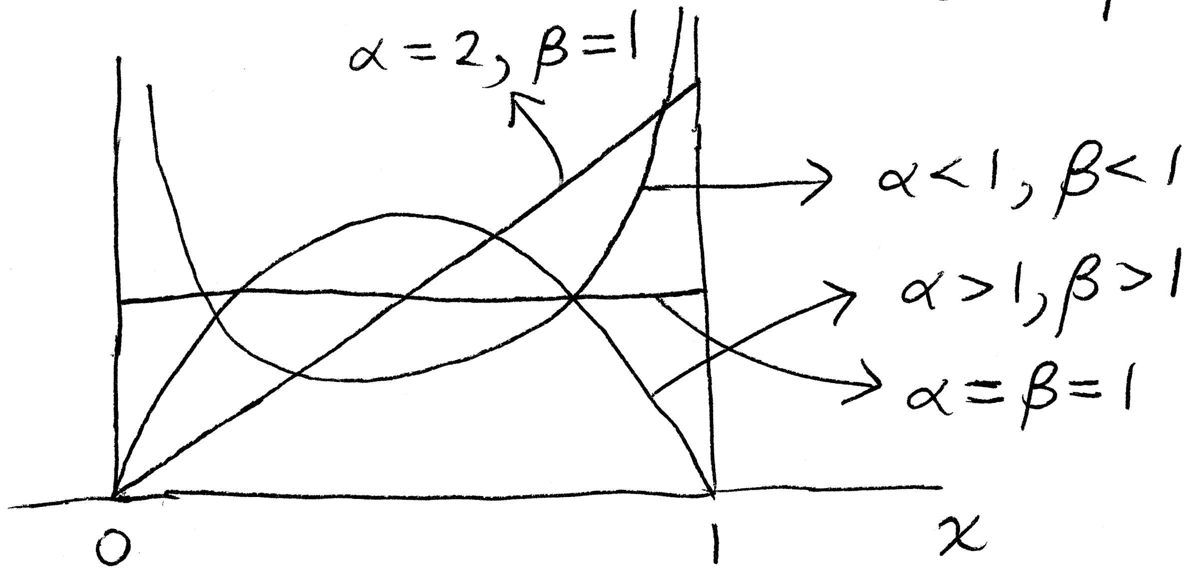
$$\alpha>1 \Rightarrow \lim_{x\to 0^+} f(x) = 0$$

$$\beta<1 \Rightarrow \lim_{x\to 1^-} f(x) = \infty$$

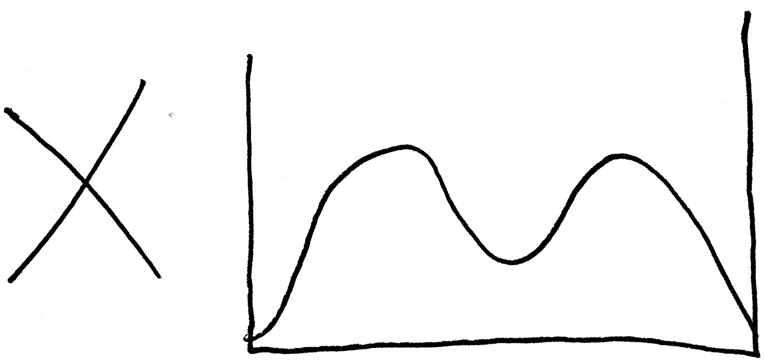
$$\beta>1 \Rightarrow \lim_{x\to 1^-} f(x) = 0$$

Setting $\alpha=\beta=1$ gives Uniform$(0,1)$ distn.

Modifying α and β gives a variety of shapes.



But you can<u>not</u> get a bimodal distribution



(unless the pdf blows up at 0 and 1 giving modes at 0 and 1)

The Beta distribution is used to model quantities taking values in $(0,1)$ (such as proportions and probabilities).

For a Beta distribution:

$$EX = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{Var}X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

which are easy consequences of

$$EX^K = \frac{\Gamma(K+\alpha)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+K)}$$

$$= \frac{(\alpha+k-1)(\alpha+k-2)\cdots\alpha}{(\alpha+\beta+k-1)\cdots(\alpha+\beta)}$$

MGF $M_X(t)$ exists for all t, (since X is bounded)

but is not useful. (It does <u>not</u> have a simple closed form.)