Bivariate Transformations

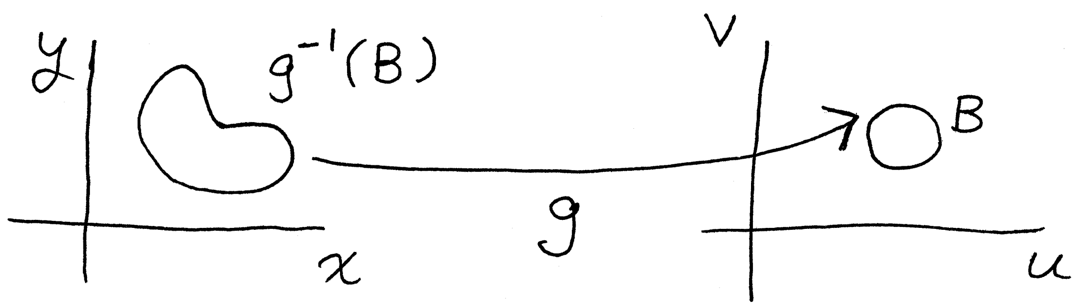
(U,V) is a function of (X,Y).

$(U,V) = g(X,Y)$ where $g: \mathbb{R}^2 \to \mathbb{R}^2$.

Given the distn. of (X,Y),

find the distn. of (U,V).

First principles



Define $g^{-1}(B) = \{(x,y): g(x,y) \in B\}$

for any set $B \subset \mathbb{R}^2$.

Then $P((U,V) \in B) = P((X,Y) \in g^{-1}(B))$.

Formula for joint density (in a special case)

Suppose:

① (X,Y) has a joint pdf $f_{X,Y}(x,y)$.

② g is 1-1 and "smooth"

from $\mathcal{A} = \{(x,y) : f_{X,Y}(x,y) > 0\}$

to $\mathcal{B}$ = image of $\mathcal{A}$ under the map g

$= \{(u,v) : (u,v) = g(x,y)$ for some $(x,y) \in \mathcal{A}\}$.

Then g has an inverse $h = g^{-1}$ and h is also smooth.

<u>Notation</u>: Let $g = (g_1, g_2)$ with

$u = g_1(x,y)$ and $v = g_2(x,y)$

and $h = (h_1, h_2)$ with

$x = h_1(u,v)$ and $y = h_2(u,v)$.

<u>Result</u>: If ① and ② are true, then (U,V) has a joint pdf given by

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J| \quad \text{for } (u,v) \in \mathcal{B}$$

where J is the Jacobian of the inverse transformation h.

The Jacobian

$$J = J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

determinant

matrix of partial derivatives

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

where $x = h_1(u,v)$,

$y = h_2(u,v)$.

Comments:

① $$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J| \quad \text{for } (u,v) \in \mathcal{B}$$

is the "vector" version of

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy} g^{-1}(y)\right| \quad \text{for } y \in \mathcal{Y}.$$

②

$f_{X,Y}(h_1(u,v), h_2(u,v))$ is simply $f_{X,Y}(x,y)$ re-expressed in terms of u and v.
Just replace x and y by expressions in terms of u and v.

Heuristic derivation of the density of $(U, V) = g(X, Y)$

Let (u, v) be a particular point in the (U, V) plane.

Suppose the function g maps (x, y) to the point (u, v) (so that $(x, y) = h(u, v)$ where $h = g^{-1}$).

Let A be a small (infinitesimal) region containing (x, y).

Suppose g maps A into the small region B containing (u, v).

Then $P((X, Y) \in A) = P((U, V) \in B)$.

We also know:

$$\begin{aligned} P((X, Y) \in A) &\approx f_{X,Y}(x, y)\,\text{Area}(A)\,, \\ P((U, V) \in B) &\approx f_{U,V}(u, v)\,\text{Area}(B) \end{aligned}$$

and the approximations can be made as accurate as we please by making the areas sufficiently small.

Equating the two expressions leads to

$$f_{X,Y}(x, y)\,\text{Area}(A) = f_{U,V}(u, v)\,\text{Area}(B)$$

so that

$$f_{U,V}(u, v) = f_{X,Y}(x, y) \cdot \frac{\text{Area}(A)}{\text{Area}(B)}\,.$$

This ratio of areas is (in the limit of areas going to zero) just the absolute value of the Jacobian $|J(u, v)|$ of the inverse transformation h, which tells us the amount of contraction or expansion produced by the mapping h at the point (u, v).

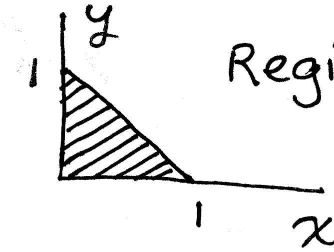
Thus we have:

$$f_{U,V}(u, v) = f_{X,Y}(h(u, v))\,|J(u, v)|$$

as desired.

Example : (X, Y) with joint pdf

$$f_{X,Y}(x,y) = 6(1-x-y)\, I_D(x,y).$$



Region D

Define $(U, V) = g(X, Y)$ by

$$U = -\log(1-X) = g_1(X,Y),$$

$$V = \frac{Y}{1-X} = g_2(X,Y).$$

Find joint pdf $f_{U,V}(u,v)$.

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① What are the supports of (X,Y) and (U,V)?

As X ranges from 0 to 1, $U = -\log(1-X)$ ranges from 0 to ∞.

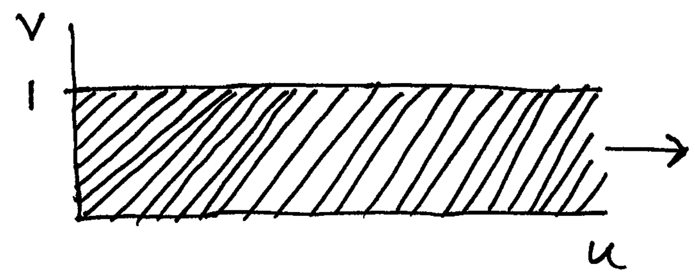
For fixed X between 0 and 1, $V = \frac{Y}{1-X}$ ranges from 0 to 1 as Y ranges from 0 to $1-X$.

Thus the supports are

$\mathcal{A} = \{(x,y) : x>0, y>0, x+y<1\} = D,$

$\mathcal{B} = \{(u,v) : 0<u<\infty, 0<v<1\}$

$= (0,\infty)\times(0,1)$ which is the infinite strip pictured below.



② Find the inverse transformation.

Solve for X, Y in
$$U = -\log(1-X), \quad (1)$$
$$V = \frac{Y}{1-X}. \quad (2)$$

(1) gives $X = 1-e^{-U} = h_1(U,V)$.

Then (2) gives
$$Y = V(1-X) = Ve^{-U} = h_2(U,V).$$

The solution is unique. Thus $g=(g_1,g_2)$ is a 1-1 and "smooth" transformation from $\mathcal{A}$ onto $\mathcal{B}$.

③ Find the Jacobian of the inverse transformation $h = (h_1, h_2)$.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{with} \quad \begin{array}{l} x = 1 - e^{-u} \\ y = ve^{-u} \end{array}$$

$$= \begin{vmatrix} e^{-u} & -ve^{-u} \\ 0 & e^{-u} \end{vmatrix} = (e^{-u})(e^{-u}) - (0)(-ve^{-u})$$

$$= e^{-2u} - 0 = e^{-2u} = J.$$

④ The joint pdf is now given by

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J|$$

$$= 6(1-x-y)\Big|_{\substack{x=1-e^{-u} \\ y=ve^{-u}}} \cdot |e^{-2u}| \quad \text{for } (u,v) \in \mathcal{B}$$

$$= 6(1-(1-e^{-u}) - ve^{-u})\,e^{-2u}$$

$$= 6(e^{-u} - ve^{-u})\,e^{-2u}$$

$$= 6(1-v)\,e^{-3u} \quad \text{for } 0<u<\infty,\ 0<v<1.$$

(*)

Comments:

The marginals for U and V are:

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv$$

$$= \int_0^1 6(1-v)e^{-3u}\,dv$$

$$= 6e^{-3u}\int_0^1 (1-v)\,dv = 6e^{-3u}\cdot\frac{1}{2}$$

$$= 3e^{-3u} \text{ for } 0<u<\infty$$

which is an exponential pdf with $\beta = \frac{1}{3}$,

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du$$

$$= 6(1-v)\int_0^{\infty} e^{-3u}\,du = 6(1-v)\cdot\frac{1}{3}$$

$$= 2(1-v) \text{ for } 0<v<1$$

which is a Beta pdf with $\alpha=1$, $\beta=2$.

Note that $f_{U,V}(u,v) = f_U(u)f_V(v)$ for <u>all</u> u, v (that is, for $-\infty<u<\infty$ and $-\infty<v<\infty$):

$$6(1-v)e^{-3u} I_{(0,\infty)\times(0,1)}(u,v)$$

$$= \left(3e^{-3u} I_{(0,\infty)}(u)\right)\left(2(1-v) I_{(0,1)}(v)\right).$$

Thus U and V are independent rv's.

We can see this by inspection (using the following lemma) directly from (*).

<u>Lemma</u>: If $f_{X,Y}(x,y) = c\,p(x)\,q(y)$ for $-\infty < x < \infty$ and $-\infty < y < \infty$, then X and Y are independent rv's with marginal densities found by "normalizing" $p(x)$ and $q(y)$.

Going back to (*):

$$f_{U,V}(u,v) = 6(1-v)e^{-3u} \text{ for } 0<u<\infty, 0<v<1$$
$$= c\,p(u)\,q(v) \text{ for } \underline{\text{all}}\ u, v$$

where $c = 6$, $p(u) = e^{-3u} I_{(0,\infty)}(u)$, and $q(v) = (1-v) I_{(0,1)}(v)$.

Thus U and V are independent.

If you don't like indicator functions, modify the lemma as follows.

<u>Lemma</u>: If the support of $f_{X,Y}(x,y)$ is a Cartesian product set, and on this support

$$f_{X,Y}(x,y) = c\,p(x)\,q(y),$$

then X and Y are independent.

A more precise and detailed wording:

<u>Lemma</u>: If

$$f_{X,Y}(x,y) = \begin{cases} c\,p(x)\,q(y) & \text{for } (x,y) \in A \times B \\ 0 & \text{otherwise} \end{cases}$$

then X and Y are independent

and $f_X(x) = a\,p(x)$ for $x \in A$, (0 o.w.)

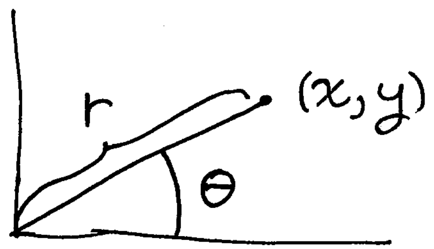
and $f_Y(y) = b\,q(y)$ for $y \in B$,

where a, b are normalizing constants.

Example :

Suppose X, Y iid $N(0,1)$.

Define (R, Θ) = polar coordinates of (X,Y).



Then $R = \sqrt{X^2 + Y^2}$

$$\Theta = \sin^{-1}\left(\frac{Y}{\sqrt{X^2+Y^2}}\right) \text{ or } = \tan^{-1}\left(\frac{Y}{X}\right).$$

Define $S = R^2$.

Find the joint pdf of (S, Θ).

X, Y iid $N(0,1)$ implies

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

(by independence)

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \quad \text{for } -\infty < x < \infty, \; -\infty < y < \infty.$$

We know that (x,y) may be obtained from the polar coordinates (r,θ) by

$$x = r\cos\theta$$
$$y = r\sin\theta .$$

Thus we have

$$S = X^2+Y^2 \quad = g_1(X,Y)$$
$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right) = g_2(X,Y)$$

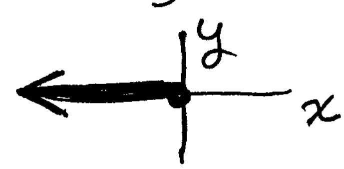
and the inverse transformation

$$X = \sqrt{S}\cos\Theta = h_1(S,\Theta)$$
$$Y = \sqrt{S}\sin\Theta = h_2(S,\Theta) .$$

The transformation $g=(g_1,g_2)$ is 1-1 and smooth from $\mathcal{A}$ to $\mathcal{B}$ where

$$\mathcal{A} = \mathbb{R}^2 - \{(x,y) : x \leq 0,\ y=0\}, \text{ and}$$
$$\mathcal{B} = \{(s,\theta) : s>0,\ -\pi<\theta<\pi\}.$$

<u>Note</u>: g is discontinuous on the ray $\{(x,y) : x\leq 0, y=0\}$ consisting of the points where $\theta = \pm\pi$.



This does not cause any problems because this ray has probability zero:

$$P\{(X,Y) \in \text{ray}\} = 0.$$

<u>The Jacobian of the inverse transformation</u>

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \qquad \text{with } x = \sqrt{s}\cos\theta, \quad y = \sqrt{s}\sin\theta$$

$$= \begin{vmatrix} \frac{\cos\theta}{2\sqrt{s}} & \frac{\sin\theta}{2\sqrt{s}} \\ -\sqrt{s}\sin\theta & \sqrt{s}\cos\theta \end{vmatrix} = \frac{1}{2}(\cos^2\theta + \sin^2\theta)$$

$$= \frac{1}{2}$$

<u>The Joint pdf</u>

$$f_{S,\Theta}(s,\theta) = f_{X,Y}(h_1(s,\theta), h_2(s,\theta))\,|J|$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \Big|_{\substack{x=\sqrt{s}\cos\theta \\ y=\sqrt{s}\sin\theta}} \cdot \frac{1}{2}$$

$$= \frac{1}{4\pi} e^{-s/2} \quad \text{for } \underbrace{s>0,\ -\pi<\theta<\pi}_{(s,\theta)\in \mathcal{B}}$$

$$= \underbrace{\left(\frac{1}{2} e^{-s/2} I_{(0,\infty)}(s)\right)}_{\substack{\text{pdf of exponential} \\ \text{with } \beta=2.}} \cdot \underbrace{\left(\frac{1}{2\pi} I_{(-\pi,\pi)}(\theta)\right)}_{\substack{\text{pdf of} \\ \text{Uniform}(-\pi,\pi).}}$$

Thus S and Θ are independent with the distributions given above.

Note: We have discovered that

$S = X^2 + Y^2 \sim$ exponential $(\beta = 2)$ which is the same as χ^2_2.

The technique we used is generally useful: To find the distn. of U = function of (X,Y), introduce another convenient random variable V = function of (X,Y), then find the joint pdf of (U,V) and integrate over V.

Distributions of Sums and Ratios

If X and Y have joint pdf $f_{X,Y}(x,y)$, then

① $Z = X+Y$ has pdf
$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx,$$

② $Z = Y/X$ has pdf
$$f_Z(z) = \int_{-\infty}^{\infty} |x|\, f_{X,Y}(x, zx)\,dx.$$

When X, Y are independent
$$f_{X,Y}(x, z-x) = f_X(x) f_Y(z-x),$$

$$f_{X,Y}(x, zx) = f_X(x) f_Y(zx).$$

In this case ① is known as a "convolution integral" or as the "convolution of two densities".

"Formula" approach for Y/X

Define the transformation

$$\begin{aligned} U &= X \quad &(= g_1(X,Y)) \\ V &= Y/X \quad &(= g_2(X,Y)) \end{aligned}$$

with inverse transformation

$$\begin{aligned} X &= U \quad &(= h_1(U,V)) \\ Y &= UV \quad &(= h_2(U,V)) \end{aligned}.$$

The mapping $(X,Y) \mapsto (U,V)$ is 1-1 from $\mathcal{A} = \{(x,y): x \neq 0\}$ onto $\mathcal{B} = \{(u,v): u \neq 0\}$.

The Jacobian of the inverse transformation is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & v \\ 0 & u \end{vmatrix} = u$$

so that

$$f_{U,V}(u,v) = f_{X,Y}(u,uv)\,|u| \text{ for } u \neq 0 \text{ and}$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_{-\infty}^{\infty} |u|\, f_{X,Y}(u,uv)\,du.$$

Proof:

"First principles" or "Formula" approach possible.

"Formula" approach: $X+Y$

$(X,Y) \quad (U,V)$

$$\left.\begin{aligned} U &= X \\ V &= X+Y \end{aligned}\right\} \text{ many other choices}$$

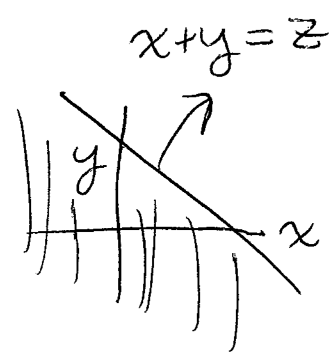
Get joint density $f_{U,V}(u,v)$.

Integrate out u to get $f_V(v)$

"First principles"

Find $F_Z(z)$ where $Z = X+Y$.

$f_Z(z) = \frac{d}{dz} F_Z(z)$.



$$\{Z \le z\} = \{X+Y \le z\}$$

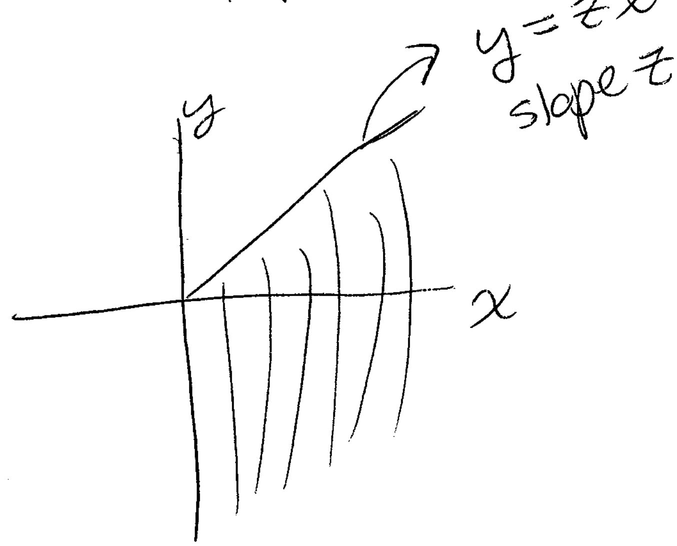
$$= \iint_A \underbrace{f_{X,Y}(x,y)}_{f_X(x) f_Y(y)} \, dx\, dy$$

First principles $\quad Z = Y/X \quad$ (Assume $f_X(x) = 0$ for $x \leq 0$)

$$F_Z(z) = P\{Z \leq z\}$$

$$= P\{Y/X \leq z\} = \iint_A f_X(x) f_Y(y)\, dx\, dy$$

$Y \leq zX$



Caution:

when applying
$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx,$$
you must think carefully about where the integrand is positive and where it is zero.

Example Suppose X, Y independent with
$$f_X(x) = \alpha e^{-\alpha x} \text{ for } x > 0$$
$$f_Y(y) = \beta e^{-\beta y} \text{ for } y > 0$$

Then $Z = X + Y$ has density
$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\substack{\text{zero} \\ \text{for } x<0}} \underbrace{f_Y(z-x)}_{\substack{\text{zero when} \\ z-x<0 \\ \text{or } x>z}}\,dx$$
$$= \int_0^z \alpha e^{-\alpha x} \beta e^{-\beta(z-x)}\,dx$$
for $z > 0$ ($f_Z(z) = 0$ for $z < 0$.)

<u>Example</u>: If X, Y iid with common density $f(x) = 6x(1-x)$, $0<x<1$, find the density of $Z = X+Y$.

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f(x)}_{\substack{\text{positive}\\ \text{for } 0<x<1}} \, \underbrace{f(z-x)}_{\substack{\text{positive for}\\ 0<z-x<1\\ \text{which means}\\ x<z \text{ and}\\ x>z-1}} \, dx$$

$$= \int_{\max(0,\, z-1)}^{\min(1,\, z)} 6x(1-x)\; 6(z-x)(1-z+x)\, dx.$$