M. S. Comprehensive Exam (Part I of Written Exam)
Saturday, January 3, 1998

You have three hours. Do as many problems as you can. No one is expected to answer all the problems correctly. Partial credit will be given. All problems are worth an equal amount of credit.

Put your solution to each problem on a separate sheet of paper.

Problem 1.

(a) I often recommend to clients using multiple regression that, if they have two predictor variables which are highly correlated, they consider using instead the sum and the difference of the variables, because their correlation will be low. Is this true?

(b) Find a condition for two linear combinations of two variables to be uncorrelated. How would you modify the consultant's advice?

Problem 2. In a multiple regression of y on x_1, x_2, x_3, and x_4, with 30 observations, suppose you wanted to test whether it was reasonable to assume β_2, the regression coefficient of x_2, equalled β_3, the coefficient of x_3.

(a) Describe how you would test this hypothesis, identifying the sums of squares and specifying the degrees of freedom.

(b) How would you test the hypothesis that $\beta_2 = 2\beta_3$?

Problem 3. Suppose that Y is a gamma random variable with the density function

$$f(y) = \begin{cases} \lambda^r y^{r-1} e^{-\lambda y}/\Gamma(r), & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases}$$

where $\lambda > 0$ and $r > 0$.

(a) Show that the moment generating function of Y is $M(t) = (\frac{\lambda}{\lambda - t})^r$ for $t < \lambda$.

(b) Find the mean $\mathrm{E}(Y)$ and the variance $\mathrm{Var}(Y)$.

Problem 4. Let X have a Poisson distribution with parameter λ. Suppose that we want to estimate $g(\lambda) = e^{-3\lambda}$.

(a) Show that $T(X) = (-2)^X$ is an unbiased estimate of $g(\lambda)$.

(b) Explain why $T(X)$ is not a good estimate of $g(\lambda)$ even though it is unbiased.

Problem 5. Let $\{X, Y\}$ be jointly distributed with joint density function

$$f(x, y) = \begin{cases} e^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{Otherwise} \end{cases}$$

(a) What is the marginal density function of X?

(b) What is the conditional density function of Y given X?

(c) What is $\mathrm{E}(Y|X = x)$?

Problem 6. Tom has α batteries in a box of which β are dead. He tests them randomly and one by one. Every time that a good battery is drawn, he will return it to the box; every time that a dead battery is drawn, he will replace it by a good one.

(a) What is the average number of good batteries in the box after n of them are checked?

(b) What is the probability that on the nth draw Tom draws a good battery?

Problem 7. In a contest, contestants A, B, and C are each asked, in turn, a general scientific question. If a contestant gives a wrong answer to a question, he goes out of the game. The remaining two will continue to compete until one of them goes out. The last person remaining is the winner. Suppose that a contestant knows the answer to a question independent of the other contestants with probability p. What is the probability that A goes out first, B next, and C wins?

Problem 8. Independent trials, each resulting in a success with probability p, are successively performed. Let N be the time of the first success. Find $\mathrm{Var}(N)$ given that $E(N) = \frac{1}{p}$.

Problem 9.

(a) If X, Y, and Z are any random variables with finite variances, and the conditional covariance of X and Y given Z is defined as

$$\mathrm{Cov}((X,Y)|Z) = E((X - E(X|Z))(Y - E(Y|Z))|Z),$$

show that

$$\mathrm{Cov}(X,Y) = E[\mathrm{Cov}((X,Y)|Z)] + \mathrm{Cov}[E(X|Z), E(Y|Z)].$$

(b) Use (a) to derive the double expectation theorem

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Z)] + \mathrm{Var}[E(X|Z)],$$

where $\mathrm{Var}(X|Z) = E((X - E(X|Z))^2|Z)$ is the conditional variance given Z.