M. S. Comprehensive Exam
Monday, August 16, 2004

You have three hours. Do as many problems as you can. No one is expected to answer all the problems correctly. Partial credit will be given. All problems are worth an equal amount of credit.

Put your solution to each problem on a separate sheet of paper.

Problem 1. You work for a company that makes paper helicopters such as the one you received with this examination. Your boss has told you to design a better paper helicopter. (Better paper helicopters land closer to the target when dropped from a height of six feet.) You may modify the weight of the paper, the length and width of all body parts, the amount of weight added to the helicopter (one or two small paper clips), the color of the paper, the type of paper, and any other parts of the helicopter that you think useful to modify. Describe how you would design and run an experiment (or a series of experiments, if you prefer) to develop a better paper helicopter.

Problem 2. The water flow from Wakulla Springs is being modeled. The response y is discharge at the opening of the springs. Hydrologists are attempting to relate this to three external factors using multiple linear regression. These three factors are

x_1 = amount of rainfall in previous 24 hours
x_2 = temperature of springs water
x_3 = level of local spray-field pumping (none, low, high)

The hydrologists assume that the only affect the pumping level has on the discharge is to change the constant (intercept) associated with the linear regression model, i.e., pumping is additive.

(a) Write out the multiple linear regression model for this analysis. Clearly label your variables.

(b) The hydrologists wish to test their assumption on pumping. Modify your answer to part (a) so that this test can be accomplished.

(c) Using the model from part (b), give the expected mean response at each of the three pumping levels, and state the null and alternative hypotheses needed to test the assumption on pumping levels.

(d) Under the assumption of normal errors with mean zero and constant variance, the correct test statistic for this test is distributed as an F random variable. What are the degrees of freedom associated with F?

Problem 3. An experiment was planned to compare three different fertilizers (A, B, C) on water melon yields. The treatments were randomly assigned according to a Latin square design conducted over a large farm plot which was divided into rows and columns. The watermelon yields (in tons per acre) were recorded below after the growing season.

	Column		
Row	1	2	3
1	9.5 (B)	6.8 (C)	4.9 (A)
2	7.9 (A)	9.1 (B)	6.6 (C)
3	5.6 (C)	7.6 (A)	8.7 (B)

(a) If you were the experimenter, how would you randomize the design? Write an appropriate additive model for the data and state your assumptions on the model.

(b) Conduct an analysis of variance ($\alpha = 0.05$) based on the data and draw your conclusions.

Problem 4. Let X and Y be independent Gaussian random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively.

(a) Use moment generating functions to *derive* the distribution of $Y - X$. Quote all the theoretical results that you are using.

(b) If $\mu_1 = 50.4$, $\mu_2 = 47.2$, $\sigma_1^2 = 37.7$ and $\sigma_2^2 = 36.06$, compute $P(Y \geq X)$.

Hint: You can use without proof that the moment generating function of a Gaussian random variable with mean μ and variance σ^2 is $M(t) = e^{t\mu}e^{t^2\sigma^2/2}$.

Problem 5. Let X_1 and X_2 be two independent random variables with density functions $f_1(x)$ and $f_2(x)$, respectively.

(a) Derive the probability density function of the random variable $Y = X_1 + X_2$. (If you know the result but cannot prove it, just state it for partial credit and go on to the next part.)

(b) Using this result, find the density function of Y, when X_1 and X_2 are two exponential random variables with intensities λ_1 and λ_2, respectively (that is, the means are $1/\lambda_1$ and $1/\lambda_2$, respectively).

Problem 6. Suppose X and Y are nonnegative, independent, continuous random variables such that

$$P(Y > z) = (P(X > z))^{\beta}$$

for all $z \geq 0$ where β is a positive constant. Find $P(X \leq Y)$.

Problem 7. Suppose you observe a single observation X from the Poisson distribution with the pmf:

$$f(x \mid \theta) = \frac{\theta^x e^{-\theta}}{x!}.$$

where the unknown parameter $\theta > 0$.

(a) Find the MLE of $e^{-\theta}$.

(b) Is the MLE an unbiased estimator of $e^{-\theta}$?

(c) Can you come up with an unbiased estimator of $e^{-\theta}$?

Problem 8. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables having an exponential distribution with mean β.

(a) Find the maximum likelihood estimator (MLE) of β and *derive* its asymptotic (large sample) distribution.

(b) Find the method of moments estimator (MOM) of β.

(c) Let X be a new independent observation from the same exponential distribution that generated the data $X_1, X_2, \ldots, X_n$. Assume that for these n data points we observed $\bar{X} = 3.4$. Use the estimate of β from either part (a) or (b) of the problem to approximate $P(X \leq 1)$.

Problem 9. Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli(p) where $n \geq 2$. What is the best unbiased estimate of $p^{\mathbf{2}}$? Justify your answer.