Qualifying Exam, Part II
Friday, January 4, 1991

You have four hours. Do as many problems as you can. No one is expected to answer all the problems correctly. Partial credit will be given.

Please put each problem on a separate sheet of paper.

Problem 1: At Tall Timbers Research Station a biologist had the following problem. There were 24 plots; 6 had been burned yearly; 6 had been burned every other year; 6 had been burned every three years; 6 had been burned every four years. The researcher had calculated the number of plant species in common for every pair of plots, 276 pairings in all. She wanted to test whether plots which had been burned the same number of times had more species in common than plots which had been burned a different number of times. What kind of test would you devise? Assume that computers exist, but be specific. (A one-word answer is not acceptable.)

Problem 2: Chemist #1 runs an experiment to compare effects of two different catalysts and a control (no catalyst) on yield of a desired ingredient. She does five runs in each of the three treatment groups. Chemist #2 does the same experiment (same two catalysts plus control, five runs per group). The table below gives the within-group means and sums of squares, respectively, for Chemist #1, Chemist #2, and Chemists #1 and #2 combined. Test the null hypothesis that the data can be viewed as coming from a single experiment, that is, that the data from the two different experiments follow the same one-way ANOVA model.

	No Catalyst	Catalyst A	Catalyst B
Chemist #1	5, 6	10, 6	14, 6
Chemist #2	4, 6	8, 6	13, 6
#1 and #2 combined	4.5, 14.5	9, 22	13.5, 14.5

Tabled values are $\bar{X}$, $\sum(X_i - \bar{X})^2$.

[You are not given any statistical tables. You should clearly indicate what tabled value you need (specify the distribution, tail probability, degrees of freedom, etc.) and how it is to be used.]

Problem 3: Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the Poisson distribution with mean λ where $\lambda > 0$ is unknown. Let $\theta = e^{-\lambda}$.

(a) Find $\hat{\theta}_n$, the uniformly minimum variance unbiased estimator of θ.

(b) Find the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \to \infty$.

Problem 4: Let $X_1, X_2, \ldots$ be i.i.d. with density

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ |x|^{-3} & \text{if } |x| > 1. \end{cases}$$

Prove that

$$(n \log n)^{-1/2} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} N(0,1) \,.$$

(Hint: truncate the X_i's at $\pm\sqrt{n}\,$.)

Problem 5: Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ and we know with certainty that either $\sigma^2 = 1$ or $\sigma^2 = 4$. Define $\theta = P(X_1 > 3)$.

(a) Find the MLE $\hat{\theta}_n$ of θ.

(b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \to \infty$?

Problem 6: Let $\{X_n\}$ be a sequence of random variables.
(a) Show that $X_n \to 0$ in probability if and only if

$$E\left(\frac{1}{1+X_n^2}\right) \to 1 \,.$$

Now suppose that

$$\sum_{n=1}^{\infty} \left(E\left(\frac{1}{1+X_n^2}\right) - 1 \right) > -\infty \,.$$

(b) Show that $X_n \to 0$ with probability 1.

(c) In fact, show that $\sum X_n^2 < \infty$ with probability 1.

Problem 7: Let $X_1, X_2, \ldots$ be a sequence of i. i. d. random vectors in $\mathbf{R}^2$ such that

$$P[X_i = (1,0)] = P[X_i = (0,1)] = \tfrac{1}{4} = P[X_i = (-1,0)] = P[X_i = (0,-1)]$$

Let $S_n = X_1 + \cdots + X_n$. *Without using the multivariate central limit theorem,* show that $S_n/\sqrt{n}$ converges to the bivariate normal distribution with mean $(0, 0)$ and covariance matrix $\frac{1}{2}\mathbf{I}$.

Problem 8: Consider the general linear model: $Y = X\beta + e$ where Y and e are $n \times 1$ random vectors, β is a $p \times 1$ vector of parameters, and X is a $n \times p$ $(n > p)$ matrix of constants. Assume that the rank of X is k $(k < p)$. Further, assume that $e \sim N_n(0, \sigma^2 I_n)$, $\sigma^2 > 0$.

(a) Let H be a $p \times m$ $(p \geq m)$ known matrix such that $H'\beta$ is estimable. If $\hat{\beta}$ is a least squares estimate of β, show that $H'\hat{\beta}$ is *independent* of $Q = (Y - X\hat{\beta})'(Y - X\hat{\beta})$.

(b) Show that $H'\hat{\beta} \sim N_m(H'\beta, \sigma^2 H'(X'X)^- H)$ and that $Q \sim \sigma^2\chi^2_{n-k}$, where A^- denotes a generalized inverse of A.