# Ph. D. Qualifying Exam
# Monday, January 4, 2010

Please submit solutions to at most **eight** problems, including only one of the two STA 5208 problems. You have five hours. No one is expected to answer all the problems correctly. Partial credit will be given. All problems are worth an equal amount of credit.

**Put your solution to each problem on a separate sheet of paper.**

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*Applied Statistics*

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**Problem 1.** (5166) In a randomized block experiment, consider the following linear model:

$$Y_{ti} = \mu + \tau_t + \beta_i + \epsilon_{ti}, \quad t = 1, \ldots, k; \;\; i = 1, \ldots, n,$$

where $\mu$ is the overall mean, $\tau_t$ is the effect corresponding to the $t$th treatment, and $\beta_i$ is the effect for the $i$th block. The $\epsilon_{ti}$'s are assumed to be iid $N(0, \sigma_\epsilon^2)$ variables.

**(a)** Let $\bar{Y}_{t\cdot} = \frac{1}{n}\sum_{i=1}^n Y_{ti}$, $\bar{Y}_{\cdot i} = \frac{1}{k}\sum_{t=1}^k Y_{ti}$, and $\bar{Y} = \frac{1}{nk}\sum_{t=1}^k \sum_{i=1}^n Y_{ti}$. Calculate the expected values and variances of $\bar{Y}_{t\cdot}$, $\bar{Y}_{\cdot i}$, and $\bar{Y}$. What is the correlation between $\bar{Y}_{t\cdot}$ and $\bar{Y}_{\cdot i}$?

**(b)** Let $S_B = k\sum_{i=1}^n (\bar{Y}_{\cdot i} - \bar{Y})^2$, $S_T = n\sum_{t=1}^k (\bar{Y}_{t\cdot} - \bar{Y})^2$, and $S_D = \sum_{t=1}^k \sum_{i=1}^n (Y_{ti} - \bar{Y})^2$. Find the mean values of $S_B$, $S_T$, and $S_D$.

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**Problem 2.** (5167) Consider the linear regression model:

$$Y = X\boldsymbol{\beta} + \xi,$$

where $Y = (y_1, \ldots, y_n)'$, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)'$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ and $X$ is an $n \times p$ full-rank matrix. The process $\{\xi_i\}$ is generated by the autoregressive model:

$$\xi_i - \phi\xi_{i-1} = \epsilon_i,$$

where $\{\epsilon_i, i = \ldots, -1, 0, 1, \ldots,\}$ are iid $N(0, \sigma^2)$ variables. Let $\hat{\boldsymbol{\beta}}$ be the least squares estimate of $\boldsymbol{\beta}$. Define $\hat{Y} = X\hat{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\xi}} = Y - \hat{Y}$.

**(a)** Find the covariance matrix of $\boldsymbol{\xi}$. What are means and covariance matrices of $\hat{Y}$ and $\hat{\boldsymbol{\xi}}$? What are the distributions of $\hat{Y}$ and $\hat{\boldsymbol{\xi}}$?

**(b)** Are $\hat{Y}$ and $\hat{\boldsymbol{\xi}}$ independent? Show your reasons.

**(c)** Can you find an estimate for $\boldsymbol{\beta}$, say $\tilde{\boldsymbol{\beta}}$, such that $\tilde{Y} = X\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\xi}} = Y - \tilde{Y}$ are independent?

**Problem 3.** (5168) In a generalized linear model (GLM), the response variable $Y$ is assumed to have a density function with the exponential dispersion form:

$$f(y;\theta,\phi) \quad = \quad \exp\{[y\theta - b(\theta)]/a(\phi) + c(y,\phi)\}.$$

**(a)** Suppose a random variable $Y \sim \text{Poisson}(\mu)$. Express the Poisson mass function in the exponential form in terms of the canonical parameter $\theta = \log(\mu)$. Derive the deviance $D(y,\mu)$ for the Poisson model.

**(b)** Suppose that $\{Y_i \sim \text{Poisson}(\mu_i), i = 1,\ldots,n\}$ are independent random variables and the mean values $\boldsymbol{\mu} = (\mu_1,\mu_2,\ldots,\mu_n)'$ follow a log-linear model. Let $H_0 \subset H_1$ be nested hypotheses (fewer covariates under $H_0$). Show that the corresponding model deviances satisfy the Pythagorean relationship

$$D(\boldsymbol{y},\hat{\boldsymbol{\mu}}^{(0)}) = D(\boldsymbol{y},\hat{\boldsymbol{\mu}}^{(1)}) + D(\hat{\boldsymbol{\mu}}^{(1)},\hat{\boldsymbol{\mu}}^{(0)}),$$

where $\boldsymbol{y} = (y_1,\ldots,y_n)'$ is the vector of observed Poisson frequencies, and $\hat{\boldsymbol{\mu}}^{(0)}$ and $\hat{\boldsymbol{\mu}}^{(1)}$ are the vectors of fitted mean values under $H_0$ and $H_1$, respectively.

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**Problem 4.** (5507) Suppose we have 8 observations from $X$ and 9 observations from $Y$. We would like to apply the Wilcoxon rank-sum test to the data to see if there is any treatment effect, i.e., $H_0 : \theta_x = \theta_y$ vs. $H_1 : \theta_x \neq \theta_y$, where $\theta_x$ and $\theta_y$ are the location parameters of $X$ and $Y$ respectively. Let $W$ be the Wilcoxon rank-sum statistic for the sample $Y$. Assume there are no ties in the data.

**(a)** Under $H_0$, calculate $P(W = 117)$.

**(b)** Suppose you reject $H_0$ if $W = 45$ or if $W = 117$, and you accept $H_0$ otherwise. If you are testing $H_0$ at level $\alpha$, what are the possible values of $\alpha$ which lead to this particular rejection region?

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**Problem 5.** (5707) Observations on two responses are collected for three groups. The observation vectors $\binom{x_1}{x_2}$ are

Group 1: $\binom{7}{6}, \binom{4}{9}, \binom{6}{3}, \binom{5}{8}, \binom{8}{4}$
Group 2: $\binom{4}{10}, \binom{5}{7}, \binom{8}{8}, \binom{3}{11}$
Group 3: $\binom{6}{7}, \binom{6}{6}, \binom{9}{5}$

**(a)** Evaluate Wilks lambda statistic $\Lambda^*$.

**(b)** Test the equality of the mean vectors in the three groups at level $\alpha = 0.05$. What are your assumptions?

**Note:** An $F$-Table is attached to the end of this exam

**Problem 6.** (5856)

**(a)** Assume the time series $\{X_t\}$ is an autoregressive (AR) process

$$(1 - B + 0.5B^2)(1 + 1.6B + 0.8B^2)X_t = a_t,$$

where $B$ is the backward shift operator and $\{a_t\}$ is a white noise process. Check the **stationarity** condition. Then calculate the autocorrelation $\rho_k$ for $k = 0, 1, 2, 3$.

**(b)** Assume the time series $\{X_t\}$ is a moving average (MA) process

$$X_t = 1 + a_t - 1.2a_{t-1} + 0.8a_{t-2},$$

where $\{a_t\}$ is a white noise process. Check the **invertibility** condition. Then calculate the autocorrelation $\rho_k$ for $k = 0, 1, 2, 3$.

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*Probability*

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**Problem 7.** (5446)

**(a)** Show that for any random variable $Y$ and any constant $A > 0$ we have

$$\int_{\{|Y|>A\}} |Y|dP \le \left( \int Y^2 dP \right)^{1/2} \times (P\{|Y| > A\})^{1/2}.$$

**(b)** Let $\{Z_n\}_{n\ge1}$ be a sequence of random variables such that:

$$\mathbb{E}Z_n = 0,\ \mathbb{E}Z_n^2 = \sigma^2 < \infty, \quad \text{for all } n \ge 1, \quad \text{and} \ \mathbb{E}Z_nZ_m = 0, \ \text{for all } m \ne n.$$

Define $S_n = Z_1 + Z_2 + \cdots + Z_n$ . Show that the sequence $\left\{ \frac{S_n}{\sqrt{n}} \right\}_{n\ge1}$ is uniformly integrable.

*Hint.* Use the definition of uniform integrability and result (a) above for $Y = \frac{S_n}{\sqrt{n}}$ .

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**Problem 8.** (5447) Let $(X_i, Y_i)$, $1 \le i \le n$, be a sequence of random pairs that are independent and identically distributed as a generic pair $(X, Y)$ which has mean

$\mu = (0,0)$ and $2\times 2$ covariance matrix $\Sigma$ such that $\Sigma_{11} = \Sigma_{22} = 1$ and $\Sigma_{12} = \Sigma_{21} = \rho$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$. Define the following quantities:

$$
\begin{aligned}
S_{XY} &= \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y}), \\
S_X &= \left(\sum_{i=1}^n (X_i - \bar{X})^2\right)^{1/2}, \\
S_Y &= \left(\sum_{i=1}^n (Y_i - \bar{Y})^2\right)^{1/2}, \\
\widehat{\rho}_n &= \frac{S_{XY}}{S_X S_Y}.
\end{aligned}
$$

Show that

$$\sqrt{n}(\widehat{\rho}_n - \rho) \to_d N(0, \tau^2),$$

and find $\tau^2$. State *all* the theoretical results that you will be using.

*Hint.* Use the multivariate central limit theorem to show first that

$$\sqrt{n}\,(S_{XY}/n - \rho, S_X/n - 1, S_Y/n - 1)$$

converges in distribution to an appropriate three dimensional normal distribution. In the course of this proof you may want to use the following fact: If a sequence is convergent in distribution then that sequence is bounded in probability. Then use the multivariate $\delta$ method to obtain the result.

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**Problem 9.** (5334) Given a random sample $X_i, i = 1, \ldots, n$ from an exponential distribution of mean $\mu$, find the limiting distribution of the ratio of the first sample quartile to the sample median $\dfrac{X_{[0.25n]:n}}{X_{[0.5n]:n}}$. Carefully state the results that you are using in your solution.

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**Problem 10.** (5326) Consider the hierarchical model

$$Y \mid \Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{Gamma}(\alpha, \beta)\,.$$

**(a)** Find the marginal distribution of $Y$.

[Hint for below: The solutions to parts (b) and (c) do **not** require the result of part (a). Recall that the Poisson($\lambda$) distribution has mean and variance equal to $\lambda$, and the Gamma($\alpha, \beta$) distribution has mean $\alpha\beta$ and variance $\alpha\beta^2$.]

**(b)** Find the mean and variance of $Y$.

**(c)** Find $\text{Cov}(Y, \Lambda)$, the covariance between $Y$ and $\Lambda$.

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**Problem 11.** (5807) Answer the following. (Parts (a) and (b) are **not** related.)

**(a)** Two white balls and two black balls are divided into two urns, labeled 1 and 2. Each urn contains two balls. At each step, one ball is selected at random from each urn and the two balls are interchanged (the ball from the first urn is placed in the second urn; the ball from the second is placed in the first). Define $X_n = i$ if, after the $n$th step, urn 1 contains $i$ white balls.

(**i**). Find the transition matrix $P$ for this Markov chain.

(**ii**). Find $P(X_2 = 1 | X_0 = 1)$.

**(b)** $B(t)$ is standard Brownian motion. For $s < t$, find $E(B(t)|B(s))$.

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**Problem 12.** (6346) Answer the following. (The two parts are **not** related.)

**(a)** Consider the probability space $([0,1], \mathcal{B}, m)$, the unit interval with Borel sets and Lebesgue measure. On this probability space, explicitly define two different random variables $X$ and $Y$ having a $N(0,1)$ distribution.

**(b)** Let $X_1, X_2, \ldots, X_n$ be iid $\text{Gamma}(\alpha, 1)$ and define $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Find (and give explicit formulas for) the first four cumulants of both $X_1$ and $Z = (\bar{X} - b)/c$ where $b \equiv E\bar{X}$ and $c^2 \equiv \text{Var}\bar{X}$. [Note: The $\text{Gamma}(\alpha, 1)$ mgf is $1/(1-t)^\alpha$.]

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*Theoretical Statistics*

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**Problem 13.** (5327) We observe $X_1, X_2, \ldots, X_n$ iid $\text{Bernoulli}(\theta)$ where $\theta$ is unknown.

**(a)** Suppose we are told $\theta \le \theta_0$ for some given value $\theta_0$. Find the MLE of $\theta$ under this restriction.

**(b)** Show that the LRT (likelihood ratio test) of $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$ rejects when $\sum_i X_i > b$ for some value $b$.

**(c)** Find the value of the conditional probability $P\left(X_1 = X_2 = 1 \,|\, \sum_{i=1}^n X_i = k\right)$ for integers $0 \le k \le n$. (You may use a relevant result without proof, but clearly state the result you are using.)

**Note: You will receive credit for only one of the following two problems.**

**Problem 14.** (5208; McGee) Answer the following.

**(a)** Assume $\boldsymbol{x}$ is a bivariate normal random variable, such that:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)$$

If we have a random sample, $\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix}$, $i = 1, \ldots, n$, of $n$ observations from this distribution, what is the maximum likelihood estimate of the ratio $\mu_1/\mu_2$? Provide an explanation for your answer.

**(b)** Classify each of the following matrices as positive definite, positive semi-definite, negative definite, or none of these:

$$1.\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad 2.\ \begin{pmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad 3.\ \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{pmatrix}$$

**(c)** What is the rank of each of the following matrices:

$$1.\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad 2.\ \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix} \qquad 3.\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

**Problem 15.** (5208; Patrangenaru) Consider an oscillation with period $2\pi$, given by $f(t) = a\cos(t - t_0)$, $t \in \mathbb{R}$, where $a$ and $t_0$ are unknown parameters. Let $Y_1, \ldots, Y_8$ be independent random variables with distributions

$$Y_i \sim \mathcal{N}(f(2\pi(i-1)/8), \sigma^2)\,,\ i = 1, \ldots, 8.$$

**(a)** Show that this is a linear model corresponding to the linear subspace $L_1$ of $\mathbb{R}^8$ spanned by the vectors

$$b_1 = \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$$

and

$$b_2 = \left(0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}\right)^T.$$

**(b)** If the mean vector is $\mu = \beta_1 b_1 + \beta_2 b_2$, find estimators $\widehat{\beta}_1, \widehat{\beta}_2$ of $\beta_1, \beta_2$ in terms of $Y_1, \ldots, Y_8$, and show that they are independent.

**(c)** Find the estimator $\tilde{\sigma}^2 = \tilde{\sigma}^2(Y)$, and its distribution.

**(d)** Find a test for the null hypothesis $t_0 = 0$ versus the alternative $t_0 \neq 0$.

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*Biostatistics*

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**Problem 16.** (5172) A group of children who lived near a lead smelter in El Paso, Texas, were identified and their blood levels of lead were measured. An exposed group of 46 children were identified who had blood-lead levels $\geq 40$ g/mL in 1972. A control group of 78 children were also identified who had blood-lead levels $< 40$ g/mL in 1972. The numbers of finger-wrist taps in the dominant hand (a measure of neurological function) were measured for all children. Stem-and-leaf and box plots of the finger-wrist tapping score for each group are given in a figure **attached to the end of this exam.**

**(a)** What procedures will you use to test whether the two groups have significantly different mean finger-wrist tapping scores? Hint: the two samples may not have equal variance.

**(b)** It seems there are some outliers present in both of the groups. What method will you use to check whether there is a single outlier? How about multiple outliers?

**(c)** How will you test whether the finger-wrist tapping scores are normally distributed using a Chi-square goodness-of-fit test?

Note: You do not need to do the actual tests using the data. Giving the procedures for the tests you will use is sufficient to get all points. However, you need to give the necessary details to show that you do know how to do it.

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**Problem 17.** (5238) The following table presents the results of four logistic models relating the specified covariates to death from coronary heart disease (CHD) among 4,673 observations. Age is age of the participant in years and sbp is the participant's systolic blood pressure in mmHg.

| Covariate | **Null Model** | | **Univariate** | | **Univariate** | | **Bivariate** | |
|---|---|---|---|---|---|---|---|---|
| | $\beta$ | $se(\beta)$ | $\beta$ | $se(\beta)$ | $\beta$ | $se(\beta)$ | $\beta$ | $se(\beta)$ |
| age | | | 0.038 | 0.006 | | | 0.029 | 0.007 |
| sbp | | | | | 0.019 | 0.003 | 0.016 | 0.003 |
| $\alpha$ | -1.237 | 0.053 | -3.180 | 0.328 | -3.796 | 0.350 | -4.895 | 0.433 |
| log likelihood | -1071.1 | | -1052.3 | | -1042.9 | | -1032.9 | |

**(a)** Which of these models best describes the data? How did you reach that conclusion?

**(b)** For the covariates age and sbp, provide an interpretation of the estimated coefficients in the bivariate model.

**(c)** For the covariates age and sbp in the bivariate model, describe two ways to test whether they are significantly different from zero.

**(d)** In the **Null Model**, what does $\frac{\exp(\alpha)}{1+\exp(\alpha)}$ estimate?

**(e)** In the data from which these coefficients were estimated, the average sbp for those 50 years of age or less is 128.8 mmHg, among those greater than 50 years of age, the average sbp is 137.1 mmHg. Similarly, among those with sbp $\leq 130$ mmHg the average age is 48.4 years, among those with sbp $> 130$ mmHg the average age is 51.3 years.

Provide a possible explanation for the change in the coefficients in the bivariate model as compared to the univariate model.

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**Problem 18.** (5244) An investigator wants to conduct a study examining whether the correlation between two laboratory measures differs in the two treatment arms. The primary outcome will involve testing the difference between two normal means.

She is willing to assume that the variance is equal for the two groups and is $\sigma^2$.

She specifies a minimum difference that she thinks is clinically important: $\Delta = \mu_1 - \mu_2$

She specifies the level of type I error she is willing to accept, $\alpha$.

Finally, she specifies $\beta$, the level of the type II error she is willing to accept (or alternatively, $1-\beta$ the power she wishes her study to have.

**(a)** After you have done the sample size calculation based on the data she provides, she says that after thinking about it, the minimum difference that she thinks is clinically important is really:

$$\Delta_1 = .5 * \Delta$$

What effect does this have on the estimated sample size?

**(b)** She later comes backs with yet another value:

$$\Delta_2 = 2 * \Delta$$

What effect does this have on the original sample size calculation.

**(c)** Suppose she decides she is willing to increase $\beta$. Does the required sample size increase or decrease?

**(d)** Suppose she decides to increase $\alpha$. Does the required sample size increase or decrease.

**(e)** Let $n_1$ and $n_0$ denote the number of patients in the treatment and control groups, respectively, and let $n = n_1 + n_0$. Because of cost and practical considerations she can randomize only a fixed total number $n$ of participants. Derive the values of $n_1$ and $n_0$ that maximize the power of the study. (Assume fixed costs, i.e., the costs for all patients is the same regardless of treatment group.)

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**Problem 19.** (5179) The shared frailty model for bivariate survival data is a generalization of the univariate proportional hazards model. The shared frailty model assumes that, given the cluster-specific random effect $W$ and the time-constant covariates $(x_1, x_2)$, the components $T_1$ and $T_2$ are independent. Further, the conditional hazard function of $T_j$ is given by $h_j(t_j \,|\, W, x_j) = h_{0j}(t) W \exp(\beta x_j)$, where $h_{0j}(t)$ is an unspecified baseline hazard function for $j = 1, 2$.

**(a)** Find the joint survival function $S(t_1, t_2 \,|\, W; x_1, x_2)$ given the frailty $W$.

**(b)** Find the (unconditional) joint survival function $S(t_1, t_2 \,|\, x_1, x_2)$ when the frailty random variable $W$ has a positive stable density with a Laplace transform $\psi(u) = E[\exp(-uW)] = \exp(-u^\alpha)$.

**(c)** Show that the marginal hazard $h_j(t \,|\, x_j) = P[T_j < t + dt \,|\, t < T_j; x_j]$ is given by, $h_j(t \,|\, x_j) = \alpha \exp(\alpha\beta x_j) h_{0j}(t)(H_{0j}(t))^{\alpha-1}$, where $H_{0j}(t) = \int_0^t h_{0j}(u)\, du$.

**(d)** Using (c), show that the marginal hazard $h_j(t \,|\, x_j)$ is also of proportional hazards structure.

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*Computational Statistics*

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**Problem 20.** (5106) We will analyze the use of Monte Carlo methods for estimating the quantity $\theta = \int_0^1 e^x dx = E[e^U]$, where $U$ is a $U[0, 1]$ random variable.

**(a)** Take the classical Monte Carlo estimator $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} e^{U_i}$, $U_i \sim U[0,1]$ are independent samples. Derive an expression for the variance of this classical estimator and evaluate its numerical value.

**(b)** Next, define a new random variable:

$$Y_i = \frac{e^{U_i} + e^{1-U_i}}{2} \ ,$$

and derive an expression for the variance of $Y_i$.

**(c)** Define a modified Monte Carlo estimator according to:

$$\tilde{\theta}_n = \frac{2}{n}\sum_{i=1}^{n/2} Y_i \ ,$$

and derive an expression for the variance of this modified estimator. Also, evaluate its numerical value and compare it with the variance of the classical estimator.

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**Problem 21.** (5107) In this problem we will derive the update step of the sequential Monte Carlo method.

**(a)** If $X$ and $Y$ are two random variables with $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x|y)$ being their marginal and a conditional density, respectively. If we have algorithms to generate samples from the marginal $f_Y$ and the conditional $f_{X|Y}(x|y)$, suggest a method to sample from the marginal $f_X(x)$. Prove that this method samples from the desired density.

**(b)** With a slight change of notation, we apply this idea to generate samples from a prediction density in the sequential Monte Carlo method. Given that we have samples from the the past posterior $f(X_t|Y_1,\ldots,Y_t)$ and can sample from the one-step conditional $f(X_{t+1}|X_t)$, suggest an algorithm to sample from the prediction density $f(X_{t+1}|Y_1,\ldots,Y_t)$.

**F-Table for $\alpha = 0.05$:** degrees of freedom of the numerator are column names, and degrees of freedom of the denominator are row names.

F - Table for $\alpha = 0.05$.

| $\frac{d_1}{d_2}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 |