Ph.D. Qualifying Exam
Friday–Saturday, January 6–7, 2017

Put your solution to each problem on a separate sheet of paper.

Problem 1. (5106) Let $X_1, X_2, \cdots, X_n$ be a sequence of i.i.d. observations from a logistic distribution with the probability density function

$$f(x|\theta) = \frac{\exp(-(x-\theta))}{(1+\exp(-(x-\theta)))^2}.$$

Our goal is to find the maximum likelihood estimate (MLE) of θ with the observations $\{X_i\}_{i=1}^n$.

(a) Derive an expression for the log-likelihood function

$$l(\theta) = \sum_{i=1}^n \log(f(X_i|\theta)),$$

such that the MLE is given by

$$\hat{\theta} = \text{argmax}_\theta \, l(\theta).$$

(b) Find the expressions for $\dot{l}(\theta)$ and $\ddot{l}(\theta)$, the first and second derivatives of l with respect to θ. Verify that $\ddot{l}(\theta) < 0$.

(c) Write out the Newton-Raphson algorithm to find the root of $\dot{l}(\theta)$.

Problem 2. (5106) Let Y be a continuous random variable with probability density function:

$$Y \sim \alpha_1 f_1(y; \mu_1, \sigma_1^2) + \alpha_2 f_2(y; \mu_2, \sigma_2^2),$$

where f_1 and f_2 are two Gaussian density functions with means μ_1, μ_2 and variances σ_1^2, σ_2^2, respectively. Also, $0 \leq \alpha_1, \alpha_2 \leq 1$, such that $\alpha_1 + \alpha_2 = 1$. Given n i.i.d. observations $\{Y_i\}_{i=1}^n$, our goal is to find the maximum likelihood estimate of

$$\theta = (\alpha_1, \mu_1, \sigma_1, \alpha_2, \mu_2, \sigma_2).$$

(a) Use the EM algorithm for iteratively estimating θ. Let $\theta^{(m)}$ be the current values of the unknown. Derive the mathematical formula to update for $\theta^{(m+1)}$.

(b) Let $L(\theta)$ denote the likelihood with parameter θ. Prove that

$$L(\theta^{(m+1)}) \geq L(\theta^{(m)}).$$

Problem 3. (5166) Consider the following linear model for a randomized block design:

$$y_{ti} = \mu + \beta_i + \tau_t + \epsilon_{ti}\,,\ t = 1, \ldots, k;\ i = 1, \ldots, n,$$

where μ is an overall mean, τ_t is the effect of tth treatment, β_i is the effect of ith block, $\{\epsilon_{ti} : t = 1, \ldots, k; i = 1, \ldots, n\}$ are assumed to be i.i.d. $N(0, \sigma^2)$.

(a) The least squares estimate of β_i is $\hat{\beta}_i = \bar{y}_{\cdot i} - \bar{y}_{\cdot\cdot}$. Find the expectation and variance of $\hat{\beta}_i$.

(b) Show the decomposition of variation for the experiment: $S_D = S_B + S_T + S_R$ where

- S_D: Total Variation of the observations,
- S_B: Sum of Squares for Blocks,
- S_T: Sum of Squares for Treatments,
- S_R: Sum of Squares for Experimental Errors.

(c) Find the expectation of S_B.

Problem 4. (5166) Consider the following unbalanced one-way random-effects model:

$$y_{ij} \;=\; \mu + \alpha_i + \epsilon_{ij}, \qquad i = 1, \ldots, k;\quad j = 1, \ldots, n_i,$$

where $\{\alpha_i, i = 1, \ldots, k\}$ are i.i.d. $N(0, \sigma_\alpha^2)$, $\{\epsilon_{ij}, i = 1, \ldots, k; j = 1, \ldots, n_i\}$ are i.i.d. $N(0, \sigma_\epsilon^2)$, and the α_i's and ϵ_{ij}'s are independent. Define

$$\text{SSA} = \sum_{i=1}^{k} n_i(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2, \qquad \text{SSE} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i\cdot})^2,$$

where

$$y_{i\cdot} = \sum_{j=1}^{n_i} y_{ij}\,,\quad \bar{y}_{i\cdot} = \frac{y_{i\cdot}}{n_i}\,; \qquad y_{\cdot\cdot} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}\,,\quad \bar{y}_{\cdot\cdot} = \frac{y_{\cdot\cdot}}{\sum_{i=1}^{k} n_i}.$$

(a) Find $\text{Cov}(y_{ij}, y_{i'j'})$. What is the distribution of $\bar{y}_{\cdot\cdot}$?

(b) Show that $\text{SSA} = \sum_{i=1}^{k} \frac{y_{i\cdot}^2}{n_i} - \frac{y_{\cdot\cdot}^2}{\sum_{i=1}^{k} n_i}$. Show that SSA and SSE are independent.

(c) Find unbiased estimates for the variances σ_α^2 and σ_ϵ^2.

Problem 5. (5167) Suppose that $\{(x_1, y_1), \dots, (x_n, y_n)\}$ is a sample from a bivariate normal distribution, i.e.,

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right), \ i = 1, \cdots, n.$$

(a) Show that the conditional distribution of y_i given x_i is normal and

$$y_i|x_i \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_i - \mu_1), \ \sigma_2^2(1 - \rho^2)\right), \ i = 1, \cdots, n.$$

(b) Define

$$\beta_1 = \rho\frac{\sigma_2}{\sigma_1}, \quad \beta_0 = \mu_2 - \beta_1\mu_1, \quad \sigma^2 = \sigma_2^2(1 - \rho^2). \tag{1}$$

Then y_i given x_i follows the simple regression model:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \ i = 1, \cdots, n,$$

where $\{\epsilon_i, \ i = 1, \cdots, n\}$ are i.i.d. $N(0, \sigma^2)$. The moment estimates of β_0, β_1, and σ^2, denoted by $\tilde{\beta}_0$, $\tilde{\beta}_1$, and $\tilde{\sigma}^2$, respectively, are obtained by simply substituting the sample means ($\bar{x}$ and $\bar{y}$), sample variances ($\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$), and sample correlation $\hat{\rho}$ into (1). Are the moment estimates the same as the least squares estimates $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\sigma}^2$?

Problem 6. (5167) Consider the linear regression model:

$$Y = X\boldsymbol{\beta} + \boldsymbol{\xi},$$

where $Y = (y_1, \dots, y_n)'$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)'$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ and X is an $n \times p$ full-rank matrix. The process $\{\xi_i\}$ is generated by the moving-average model:

$$\xi_i = \epsilon_i - \theta_1\epsilon_{i-1} - \theta_2\epsilon_{i-2},$$

where $\{\epsilon_i, i = -1, 0, 1, \dots, n\}$ are i.i.d. $N(0, \sigma^2)$ variables.

(a) Calculate the variance-covariance matrix of $\boldsymbol{\xi}$. Describe how to use the least squares method, the weighted least squares method, and the maximum likelihood method to estimate the coefficients $\boldsymbol{\beta}$ in the model. Give details of the three procedures.

(b) Suppose that $\theta_1 = 0.5$ and $\theta_2 = -2$. Let $\hat{\boldsymbol{\beta}}$ be the weighted least squares estimate of $\boldsymbol{\beta}$. Give the expression of $\hat{\boldsymbol{\beta}}$ in this case. Define $\hat{Y} = X\hat{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\xi}} = Y - \hat{Y}$. Are $\hat{Y}$ and $\hat{\boldsymbol{\xi}}$ independent? Show your reasons.

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Problem 7. (5326) Suppose that a Geiger counter is turned on at time zero and that clicks on this Geiger counter occur according to a Poisson process with a rate of λ clicks per second.

(a) Let S_t denote the random number of clicks during the time interval $(0, t)$. State a formula for $P(S_t = k)$ valid for nonnegative integers k. (No proof is needed. Just state the answer.)

(b) Let $T_1 < T_2 < T_3 < \cdots$ be the times of the first click, second click, third click, $\ldots$, and $X_1 = T_1$, $X_2 = T_2 - T_1$, $X_3 = T_3 - T_2$, $\ldots$ be the times between clicks (the interarrival times). What is the joint distribution of $X_1, X_2, X_3, \ldots$? In particular, give an explicit formula for the joint density of (X_1, X_2, X_3). (No proof is needed. Just state the answers.)

(c) Use the facts in parts (a) and (b) **or any other approach** to prove that

$$\int_t^\infty \frac{\lambda^r}{\Gamma(r)} z^{r-1} e^{-\lambda z}\, dz = \sum_{y=0}^{r-1} \frac{(\lambda t)^y e^{-\lambda t}}{y!}, \quad r = 1, 2, 3, \ldots$$

where $\lambda > 0$ and $t > 0$. Give a detailed argument.

(d) Let S_t be as defined in (a) for any value of t. Find $P(S_2 \leq 1, S_5 \leq 2, S_{10} = 3)$.

Problem 8. (5326) Suppose the random variables (X, Y) have the joint density

$$f(x, y) = x^2 y e^{-x(y+1)} \quad \text{for } x > 0, y > 0 \quad (\text{and } f(x, y) = 0 \text{ otherwise}).$$

Answer the following. Carefully specify the support of any density or joint density.

(a) Find the marginal density of Y.

(b) Find the joint density of (U, V) where $U = X(Y + 1)$ and $V = X$.

(c) Find the density of $U = X(Y + 1)$.

Problem 9. (5327) Let $X_1, X_2, \ldots, X_n$ be i.i.d. from a distribution with pdf given by

$$f(x \mid \theta) = \begin{cases} \theta^{-1} x^{(1-\theta)/\theta} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is the unknown parameter.

(a) Show that $T = -2\sum_{i=1}^{n} \log(X_i)$ is a minimal sufficient statistic for θ.

(b) Find the distribution of $Y = -2 \log X_1$.

(c) Using Basu's theorem or otherwise, find $\mathbb{E}[Y \mid T]$, the conditional expectation of the random variable Y given T.

Problem 10. (5327) Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed Poisson(λ) random variables for an unknown parameter $\lambda > 0$. Instead of observing the random variables X_i, we only observe the events $X_i = 0$ or $X_i > 0$ for $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimate (MLE) of λ and discuss when the MLE is not finite.

(b) Compute the probability that the MLE is not finite based on a sample of size n, assuming that the true value of λ is $\lambda_0 > 0$.

Problem 11. (6346) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A statement about elements $\omega \in \Omega$ is true locally almost everywhere (μ) if

i. the statement is true for all $\omega \in A$, where $A \in \mathcal{F}$, and

ii. $\mu(A' \cap F) = 0$ for any $F \in \mathcal{F}$ with $\mu(F) < \infty$.
(Note: A' is the complement of A.)

(a) Show that if something is true almost everywhere (μ) then it is true locally almost everywhere (μ).

(b) If μ is σ-finite, show that if something is true locally almost everywhere (μ) then it is true almost everywhere (μ).

Problem 12. (6346) Let $(\Omega = [0,1], \mathcal{F} = \mathcal{B}[0,1], \mu)$ be a probability space where μ is Lebesgue measure on $[0,1]$ and $\mathcal{B}[0,1]$ is the restriction of the Borel σ-field to $[0,1]$. Let X_n be a sequence of random variables given by

$$X_n(\omega) = \begin{cases} n^2, & \omega \in [0, 1/n], \\ 0, & \omega \in (1/n, 1]. \end{cases}$$

(a) Show that $X_n \xrightarrow{P} X$ by using the definition of convergence in probability, and give X.

(b) Show that $X_n \to X$ a.s. for the same X as in part (a).

(c) Prove or disprove: For $p \geq 1$, $X_n \xrightarrow{L^p} X$ for the same X as in parts (a) and (b).