Ph. D. Qualifying Exam (Part II of Written Exam)
Friday, January 4, 2002

You have four hours. Do as many problems as you can. No one is expected to answer all the problems correctly. Partial credit will be given. All problems are worth an equal amount of credit.

Put your solution to each problem on a separate sheet of paper.

Problem 1. In a Balanced Incomplete Block (BIB) design, the block size k is less than the number of treatments t. There are b blocks and each treatment is replicated r times. Furthermore, each pair of treatments is compared in λ blocks.

Consider the following linear model for a BIB design:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}, \quad i = 1, \ldots, b; \;\; j = 1, \ldots, t,$$

where α_i is the ith block effect, β_j is the jth treatment effect, and the ϵ_{ij}'s are independent $N(0, \sigma^2)$. Let $\hat{\beta}_j$ be the least squares estimate of β_j. It is known that $\mathrm{Var}(\hat{\beta}_j - \hat{\beta}_l) = \frac{2k}{\lambda t}\sigma^2$ for $j \neq l$.

(a) Show that $\lambda(t-1) = r(k-1)$.

(b) If the experiment were run as a randomized complete block design, show that $\mathrm{Var}(\hat{\beta}_j - \hat{\beta}_k) = \frac{2}{b}\sigma^2$.

(c) Show that $\frac{k}{\lambda t} > \frac{1}{b}$ and give it a statistical interpretation.

Problem 2. Consider the linear regression model:

$$Y = X\beta + \xi,$$

where $Y = (y_1, \ldots, y_n)'$, $\xi = (\xi_1, \ldots, \xi_n)'$, $\beta = (\beta_1, \ldots, \beta_p)'$ and X is an $n \times p$ full-rank matrix. The process $\{\xi_i\}$ is generated by the autoregressive model:

$$\xi_i - 0.6\xi_{i-1} = \epsilon_i,$$

where $\{\epsilon_i, i = 0, 1, \ldots, n\}$ are iid $N(0, \sigma^2)$ variables.

(a) Show that $\{\xi_i, i = 1, 2, \ldots,\}$ is a second-order stationary process. Calculate the autocovariance function $\gamma_k = \mathrm{Cov}(\xi_i, \xi_{i+k})$.

(b) How do you estimate β in this setting? Why do you choose this method? Discuss the properties of your estimate such as the mean, covariance matrix, and distribution of $\hat{\beta}$.

Problem 3. In evaluating candidates for a three-year space voyage, a psychologist evaluated four candidates, who were from Asia, Africa, South America, and Europe.

Each trainee was shown twelve pictures, in random order, and asked to rate each person on a scale of 1 to 10 on the question: *How suitable would this person be to accompany you on a long space voyage?* Naturally, the twelve pictures showed persons from Asia, Africa, South America, and Europe, three people from each group.

(a) Give an appropriate model for y, the score assigned to a picture by a candidate.

(b) Give the ANOVA table (just Source and d.f.) for this model.

(c) The psychologist thinks that trainees will tend to rate people from their own continent higher than people from other continents. How would you test this hypothesis?

(d) Show the revised Anova table.

Problem 4. Let $\{X_n\}$ be a sequence of random variables, X a random variable with distribution function F, and α a real number. Suppose $X_n \nearrow X$ on $\{X > \alpha\}$, and $X_n \leq X$ almost surely. Show that

$$\lim_{n\to\infty} E[\max(X_n, \alpha)]$$

exists and express the limit in terms of F and α.

Problem 5.

(a) State the Lévy continuity theorem.

(b) Show that if $\varphi(t)$ is a characteristic function, then so is $|\varphi(t)|^2$.

(c) Suppose that the distribution function F is infinitely divisible, i.e., for each $n \geq 1$ there exists a distribution function F_n such that F coincides with the n-fold convolution F_n^{*n}. Show that the characteristic function of F never vanishes. HINT: Apply (b) to the characteristic function of F_n and take limits.

Problem 6. Let Y_n, M_n, Z_n be the 1-st quartile, median and 3-rd quartile, respectively of $X_1, \ldots, X_n$ which are i.i.d. with common df F and pdf $f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}$. Suppose that one proposes to use M_n and $W_n = \frac{1}{2}(Y_n + Z_n)$ as estimates of θ. State the required results and obtain the asymptotic distributions of these two estimates and compute the efficiency by comparing their asymptotic variances. *Note that* $F(\theta) = \frac{1}{2}, F(\theta+1) = \frac{3}{4}$.

Problem 7. Suppose that X has a Poisson distribution with parameter θ

(a) Find the UMVUE of $\exp\{-3\theta\}$ and its corresponding variance. [Hint: evaluate $E(u^X)$ for arbitrary u.]

(b) Is the estimator found in (a) a good estimator. If not, find a better estimator.

(c) Prove that there is no unbiased estimator of $1/\theta$.

(d) Let $Y = \sum_{i=1}^{n} X_i$, where $X_1, \cdots, X_n$ is a random sample from the above Poisson distribution. We wish to estimate $1/\theta$. Consider the following estimator $T_n(Y) = n/(Y+1)$. Show that the bias of this estimator tends to 0 as $n \to \infty$ and find the asymptotic distribution of $T_n(Y)$.

Problem 8. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$.

(a) Show that $(\bar{X}, S^2)$, the sample mean and variance, is a complete sufficient statistic for (μ, σ^2).

(b) Show that $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{S} \right)^3$ is independent of $\bar{X}/S$.

Problem 9. Suppose $Y \sim N_p(a, I)$ where a is a $p \times 1$ unit vector ($a'a = 1$). Let $\Gamma = [a, \Gamma_1]$ be an orthogonal matrix (that is, Γ is a $p \times p$ orthogonal matrix whose first column is a).

(a) What is the distribution of $\Gamma' Y$? (Be specific.)

(b) What is the distribution of $Y'(I - aa')Y$? (Be specific.)

(c) Are $\Gamma' Y$ and $Y'(I - aa')Y$ independent? (Prove your answer.)