Backshift Notation

ARMA (and ARIMA) models are often expressed in "backshift" notation.

B is the "backshift operator" (also called the "lag operator"). It operates on time series, and means "back up by one time unit".

Examples:

$$Bz_t = z_{t-1} \qquad\qquad Ba_t = a_{t-1}$$
$$Bz_{t-2} = z_{t-3} \qquad\qquad Ba_{t-1} = a_{t-2}$$

B^k means "backshift k times".

Examples

$$B^k z_t = z_{t-k} \qquad\qquad B^k a_{t-2} = a_{t-2-k}$$

Backshift Algebra

As an operation on time series, the backshift operator B obeys certain algebraic rules.

Let $\{X_t\}$ and $\{Y_t\}$ be time series, and b, c, d be constants.

$$
\begin{aligned}
B\,(X_t + Y_t) &= BX_t + BY_t && (= X_{t-1} + Y_{t-1}) \\
B\,c &= c && (*) \\
B\,cX_t &= cBX_t && (= cX_{t-1}) \\
B^j B^k\,X_t &= B^{j+k} X_t && (= X_{t-j-k})
\end{aligned}
$$

To understand rule $(*)$, think of the constant 'c' as standing for the constant time series $\ldots, c, c, c, \ldots$. When you backshift this series, nothing changes.

All these rules follow from the definition: $BX_t = X_{t-1}$.

These rules can be used in combination. Here are some examples:

$$B(bX_t + cY_t + d) = bX_{t-1} + cY_{t-1} + d$$

$$\begin{aligned} & B(C + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t - \theta_1 a_{t-1}) \\ = \quad & C + \phi_1 z_{t-2} + \phi_2 z_{t-3} + a_{t-1} - \theta_1 a_{t-2} \end{aligned}$$

Backshift Polynomials

A "backshift polynomial" is a polynomial involving powers of B.

Examples:

$$1 + B \qquad\qquad 2 + 3B + 4B^2$$
$$b + cB + dB^2 \qquad\qquad bB + cB^3 + dB^5$$

A backshift polynomial represents an operation performed on time series.

Examples:

$$(1 + B)Z_t = Z_t + BZ_t = Z_t + Z_{t-1}$$
$$(2 + 3B + 4B^2)Z_t = 2Z_t + 3BZ_t + 4B^2Z_t = 2Z_t + 3Z_{t-1} + 4Z_{t-2}$$
$$(b + cB + dB^2)Z_t = bZ_t + cZ_{t-1} + dZ_{t-2}$$
$$(bB + cB^3 + dB^5)Z_t = bZ_{t-1} + cZ_{t-3} + dZ_{t-5}$$

Manipulation of Backshift Polynomials

Backshift polynomials may be treated as both polynomials and as operations on time series. That is, B may be viewed as a number or as the backshift operation ($Bz_t = z_{t-1}$).

Backshift polynomials obey the usual rules of algebra. You may add them and multiply them, etc.

ARMA Processes Expressed in Backshift Notation

Example: We can write an ARMA(2,3) in backshift form as follows.

$$z_t = C + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \theta_3 a_{t-3}$$

so that

$$z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2} = C + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \theta_3 a_{t-3}$$

$$z_t - \phi_1 B z_t - \phi_2 B^2 z_t = C + a_t - \theta_1 B a_t - \theta_2 B^2 a_t - \theta_3 B^3 a_t$$

$$(1 - \phi_1 B - \phi_2 B^2) z_t = C + (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3) a_t$$

which we abbreviate as

$$\phi(B) z_t = C + \theta(B) a_t \qquad \text{where we define}$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 \text{ and}$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3$$

Summary of Backshift Notation and its Uses

Definition: $B^k z_t = z_{t-k}$ $\quad$ $B^k a_t = a_{t-k}$

ARMA(p, q) process in backshift notation:

$$(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)z_t = C+(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)a_t$$

or in mean-centered form (which eliminates C):

$$(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)\tilde{z}_t = (1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)a_t$$

If we define polynomials:

$$\begin{aligned}\phi(B) =& 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p\\ \theta(B) =& 1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q\end{aligned}$$

we may write this more briefly as

$$\phi(B)\tilde{z}_t = \theta(B)a_t\,.$$

Stationarity

An ARMA(p, q) process is **stationary** if and only if all the solutions of $\phi(B) = 0$ lie strictly outside the unit circle in the complex plane.

Notes:

The solutions of $\phi(B) = 0$ are called the "zeros" or "roots" of $\phi(B)$.

The unit circle is the circle of radius 1 about $(0, 0)$.

The modulus of a complex number $z = x + yi$ is $|z| = \sqrt{x^2 + y^2}$, the distance of (x, y) from the origin.

A point $z = x + yi$ lies strictly outside the unit circle if $|z| > 1$.

For real numbers, "modulus" is the same as "absolute value".

Checking Stationarity

AR(1) Processes:

In this case, the AR polynomial has a single zero:

$$\phi(B) = 1 - \phi_1 B = 0 \qquad \text{iff} \qquad B = \frac{1}{\phi_1}$$

This lies strictly outside the unit circle when

$$\left|\frac{1}{\phi_1}\right| = \frac{1}{|\phi_1|} > 1 \qquad \text{or} \qquad |\phi_1| < 1$$

which is the stationarity condition stated earlier.

AR(2) Processes:

The AR(2) polynomial has two roots which are found by solving the quadratic equation:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 = 0\,.$$

From the known expressions for the roots, it is possible (but not so easy) to verify that the roots are outside the unit circle iff

$$|\phi_2| < 1,\ \phi_2 + \phi_1 < 1,\ \phi_2 - \phi_1 < 1\,.$$

AR(p) Processes: For $p \geq 3$, the roots of $\phi(B) = 0$ can be found numerically.

Note: If $\phi_1 + \phi_2 + \cdots + \phi_p \geq 1$, the process is always non-stationary. (Why?)

Stationary ARMA Processes Written in MA(∞) Form

A stationary ARMA(p, q) process may be written in MA(∞) form:

$$\begin{aligned}\tilde{z}_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\ &= \textstyle\sum_{k=0}^{\infty} \psi_k a_{t-k} \qquad (\text{with } \psi_0 = 1)\end{aligned}$$

where $\psi_k \to 0$ as $k \to \infty$ and $\sum_k |\psi_k| < \infty$. The values ψ_j (eventually) decay to zero in an exponential fashion, possibly with sinusoidal oscillation.

The coefficients $\{\psi_k\}$ are called the ψ-weights.

In backshift notation, we may write

$$\tilde{z}_t = \sum_{k=0}^{\infty} \psi_k a_{t-k} = \left(\sum_{k=0}^{\infty} \psi_k B^k \right) a_t = \psi(B) a_t$$

where we define $\psi(B) = \sum_{k=0}^{\infty} \psi_k B^k$.

“Derivation”:

$$\phi(B)\tilde{z}_t = \theta(B)a_t \quad \Rightarrow \quad \tilde{z}_t = \frac{\theta(B)}{\phi(B)}a_t = \psi(B)a_t$$

where we define $\psi(B) = \dfrac{\theta(B)}{\phi(B)}$.

The ψ-weights $\{\psi_k\}$ may be found be expanding $\psi(B) = \theta(B)/\phi(B)$. This may be done in various ways (e.g., Taylor series or recursions).

We can always expand $\psi(B) = \theta(B)/\phi(B)$ as an infinite series so that even non-stationary ARMA processes can be written in MA(∞) form $\sum_k \psi_k a_{t-k}$. However, for non-stationary processes the coefficients ψ_k no longer converge to zero as $k \to \infty$, and it is not clear if the infinite sum has any mathematical meaning. But the ψ-weights $\{\psi_k\}$ obtained by expanding $\psi(B) = \theta(B)/\phi(B)$ are still useful in computing forecast variances.

Useful Fact: A stationary ARMA(p,q) process with mean zero can be expressed as

$$\frac{\theta(B)}{\phi(B)} a_t$$