Seasonal Models

Many time series exhibit seasonal variation.

For monthly data, values may tend to be similar to values from the same month a year earlier (seasonality at lag 12)

For quarterly data, values may tend to be similar to values from the same quarter a year earlier (seasonality at lag 4).

For daily data, there may be "day of the week" effects (seasonality at lag 7).

For hourly data, there may be "time of day" effects (seasonality at lag 24).

Let s be the seasonal lag. For data exhibiting seasonal variation at lag s, it is natural to include terms at lag s (or at multiples of lag s) in the model.

Some Purely Seasonal Models (rare in practice)

$$\mathrm{AR}(1)_s \equiv \mathrm{ARIMA}(1,0,0)_s \;:\quad z_t = C + \Phi_1 z_{t-s} + a_t$$

$$\mathrm{MA}(1)_s \equiv \mathrm{ARIMA}(0,0,1)_s \;:\quad z_t = C + a_t - \Theta_1 a_{t-s}$$

$$\mathrm{ARMA}(1,1)_s \equiv \mathrm{ARIMA}(1,0,1)_s \;:\quad z_t = C + \Phi_1 z_{t-s} + a_t - \Theta_1 a_{t-s}$$

$$\mathrm{AR}(2)_s \equiv \mathrm{ARIMA}(2,0,0)_s \;:\quad z_t = C + \Phi_1 z_{t-s} + \Phi_2 z_{t-2s} + a_t$$

$$\mathrm{MA}(2)_s \equiv \mathrm{ARIMA}(0,0,2)_s \;:\quad z_t = C + a_t - \Theta_1 a_{t-s} - \Theta_2 a_{t-2s}$$

etc.

The general case is denoted $\mathrm{ARIMA}(P,0,Q)_s$.

ACF/PACF of Purely Seasonal Models

The ACF and PACF of a purely seasonal ARIMA$(P, 0, Q)_s$ model are nonzero **only at lags which are multiples of** s. Along these lags, the ACF and PACF are identical to their non-seasonal counterparts.

Combined Nonseasonal and Seasonal Processes

Most series with seasonal variation also exhibit nonseasonal variation, so it is natural to include terms at both the early lags and the seasonal lags.

Such models may be constructed in a multiplicative or non-multiplicative fashion.

A non-multiplicative Nonseasonal/Seasonal AR model:

$$z_t = C + \phi_1 z_{t-1} + \phi_2 z_{t-2} + \phi_s z_{t-s} + a_t$$

This is a special case of an AR(s) model which only has nonzero coefficients (ϕ_k's) at lags 1, 2, s.

In a similar way, we can also define non-multiplcative Nonseasonal/Seasonal MA models.

Multiplicative models are most easily expressed in backshift notation.

A Multiplicative Nonseasonal/Seasonal AR model:

$$\text{ARIMA}(2,0,0)(1,0,0)_s \equiv \text{ARMA}(2,0)(1,0)_s$$
$$(1 - \phi_1 B - \phi_2 B^2)(1 - \Phi_1 B^s) z_t = C + a_t$$

Expanding the product and eliminating the backshift notation leads to

$$(1-\phi_1 B-\phi_2 B^2-\Phi_1 B^s+\phi_1\Phi_1 B^{s+1}+\phi_2\Phi_1 B^{s+2})z_t = C + a_t$$

$$z_t-\phi_1 z_{t-1}-\phi_2 z_{t-2}-\Phi_1 z_{t-s}+\phi_1\Phi_1 z_{t-s-1}+\phi_2\Phi_1 z_{t-s-2} = C + a_t$$

which upon re-arranging becomes

$$z_t = C+\phi_1 z_{t-1}+\phi_2 z_{t-2}+\Phi_1 z_{t-s}-\phi_1\Phi_1 z_{t-s-1}-\phi_2\Phi_1 z_{t-s-2}+a_t\,.$$

This is a special case of an AR($s+2$) process.

A Multiplicative Nonseasonal/Seasonal MA model:

$$\mathrm{ARIMA}(0,0,1)(0,0,2)_s \equiv \mathrm{ARMA}(0,1)(0,2)_s$$

$$z_t = C+(1-\theta_1 B)(1-\Theta_1 B^s-\Theta_2 B^{2s})a_t$$

Expanding the product and eliminating the backshift notation leads to

$$
\begin{aligned}
z_t &= C + (1 - \theta_1 B)(1 - \Theta_1 B^s - \Theta_2 B^{2s}) a_t \\
&= C + (1 - \theta_1 B - \Theta_1 B^s + \theta_1 \Theta_1 B^{s+1} - \Theta_2 B^{2s} + \theta_1 \Theta_2 B^{2s+1}) a_t \\
&= C + a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-s} + \theta_1 \Theta_1 a_{t-s-1} - \Theta_2 a_{t-2s} + \theta_1 \Theta_2 a_{t-2s-1}
\end{aligned}
$$

This is a special case of an MA$(2s + 1)$ process.

Multiplicative nonseasonal/seasonal models may have both AR and MA terms (mixed models).

When we say 'Seasonal ARMA model' we typically mean a multiplicative nonseasonal/seasonal model of the type illustrated above.

We now describe the general case.

Multiplicative Seasonal ARMA Models

Are described using the notation:

$$\text{ARIMA}(p,0,q)(P,0,Q)_s \equiv \text{ARMA}(p,q)(P,Q)_s$$

Can be written in backshift form as

$$\Phi(B^s)\phi(B)z_t = C + \Theta(B^s)\theta(B)a_t$$

where

$$\Phi(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}$$
$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$$
$$\Theta(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \ldots - \Theta_Q B^{Qs}$$
$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$$

Terminology:

$$\text{AR-polynomial} = \Phi(B^s)\phi(B)$$
$$\text{MA-polynomial} = \Theta(B^s)\theta(B)$$

Each polynomial has a seasonal and non-seasonal factor.