Invertibility: Prolog

ARMA(p,q) process: $\phi(B)\tilde{z}_t = \theta(B)a_t$

If all the roots of $\phi(B) = 0$ are strictly outside the unit circle, then the process is stationary and

$$\tilde{z}_t = \frac{\theta(B)}{\phi(B)}a_t = \psi(B)a_t = \sum_{k=0}^{\infty}\psi_k a_{t-k} \quad \text{with } \psi_k \to 0 \text{ as } k \to \infty.$$

Similarly, if all the roots of $\theta(B) = 0$ are strictly outside the unit circle, then

$$a_t = \frac{\phi(B)}{\theta(B)}\tilde{z}_t = \pi(B)\tilde{z}_t \qquad \text{where}$$

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = 1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots = 1 - \sum_{k=1}^{\infty}\pi_k B^k$$

and $\pi_k \to 0$ as $k \to \infty$.

Note that:

$$a_t = \pi(B)\tilde{z}_t \;\Rightarrow\; a_t = \tilde{z}_t - \sum_{k=1}^{\infty} \pi_k \tilde{z}_{t-k} \;\Rightarrow\; \tilde{z}_t = a_t + \sum_{k=1}^{\infty} \pi_k \tilde{z}_{t-k}$$

Putting this all together ...

Definition of Invertibility

If all the roots of $\theta(B) = 0$ are strictly outside the unit circle, then the ARMA(p, q) process is said to be **invertible**, and can be written as an AR(∞) process:

$$\tilde{z}_t = a_t + \sum_{k=1}^{\infty} \pi_k \tilde{z}_{t-k} \quad \text{with } \pi_k \to 0 \text{ as } k \to \infty.$$

Since $\tilde{z}_t \approx a_t + \sum_{k=1}^{M} \pi_k \tilde{z}_{t-k}$ for large enough M, we see ...

For an invertible process:

• z_t can be approximated as a random shock plus a linear combination of values z_{t-k} in the recent past, i.e., there is no dependence on the remote past.

• Given data $z_1, \ldots, z_n$, residuals $\hat{a}_t$ (which are estimates of the random shocks) can be obtained from

$$\hat{a}_t = \tilde{z}_t - \sum_{k=1}^{\infty} \pi_k \tilde{z}_{t-k} \quad \text{for } t = 1, \ldots, n$$

by setting $\tilde{z}_{t-k} = 0$ for $k \geq t$. These residuals will be reasonably accurate except for small values of t.

Comments on invertibility of ARMA(p, q) processes:

- Invertibility depends only on the MA coefficients $\theta_1, \ldots, \theta_q$.

- For $q = 1$, the process is invertible if $|\theta_1| < 1$.

- For $q = 2$, the process is invertible if $|\theta_2| < 1$, $\theta_2 + \theta_1 < 1$, and $\theta_2 - \theta_1 < 1$.

(From the definition of invertibility, we see that the invertibility conditions on the MA coefficients are the same as the stationarity conditions on the AR coefficients.)

Expressing the MA(1) process as an AR(∞)

For the MA(1) process, $\phi(B) = 1$ and $\theta(B) = 1 - \theta_1 B$ so that

$$\begin{aligned}
\theta(B)a_t &= \phi(B)\tilde{z}_t \qquad \text{becomes} \\
(1 - \theta_1 B)a_t &= \tilde{z}_t \qquad \text{which implies} \\
a_t &= \left(\frac{1}{1 - \theta_1 B}\right)\tilde{z}_t \\
a_t &= (1 + \theta_1 B + \theta_1^2 B^2 + \theta_1^3 B^3 + \cdots)\tilde{z}_t \\
a_t &= \tilde{z}_t + \theta_1 \tilde{z}_{t-1} + \theta_1^2 \tilde{z}_{t-2} + \theta_1^3 \tilde{z}_{t-3} + \cdots \quad \text{so that} \\
\tilde{z}_t &= a_t - \theta_1 \tilde{z}_{t-1} - \theta_1^2 \tilde{z}_{t-2} - \theta_1^3 \tilde{z}_{t-3} - \cdots
\end{aligned}$$

We can also show this by repeated substitution, as we did when showing that the AR(1) could be written as an MA(∞).

The IACF and Invertibility

The IACF of an ARMA(p, q) process is the ACF of the "dual" ARMA(q, p) process obtained by interchanging the roles of the ϕ_i's and the θ_i's.

The dual process interchanges the roles of the polynomials $\phi(B)$ and $\theta(B)$. Thus if an ARMA process is stationary, the dual process is invertible. If a process is invertible, the dual process is stationary.

Similarly, if an ARMA process is non-invertible, the dual process is non-stationary.

Thus, the sample IACF of an invertible process should resemble the sample ACF of its dual process, which is stationary; it should decay to zero at a reasonable rate (not too slowly).

Similarly, the sample IACF of a non-invertible process should resemble the sample ACF of its dual process, which is non-stationary; it should decay to zero very slowly.

Fact: If an ARIMA process is over-differenced, it becomes a non-invertible process (so that we expect the sample IACF to decay slowly).

Example: Suppose $\{z_t\}$ is a stationary ARMA(1,1) process and let $w_t = \nabla z_t = (1 - B)z_t$. Since $\{z_t\}$ is already stationary, it does not need differencing to make it stationary, and so $\{w_t\}$ is over-differenced.

We now show that $\{w_t\}$ is a non-invertible process. Since $\{z_t\}$ is a stationary ARMA(1,1) process, we can write:

$$
\begin{aligned}
(1 - \phi_1 B)\tilde{z}_t &= (1 - \theta_1 B)a_t \\
(1 - \phi_1 B)(1 - B)\tilde{z}_t &= (1 - \theta_1 B)(1 - B)a_t \\
(1 - \phi_1 B)w_t &= \left(1 - (1 + \theta_1)B + \theta_1 B^2\right) a_t
\end{aligned}
$$

From this we see that $\{w_t\}$ is an ARMA(1,2) process. Since its MA polynomial can be factored as $(1 - \theta_1 B)(1 - B)$, it has a root of $B = 1$ which is on the boundary of the unit circle, *not* strictly outside. Therefore $\{w_t\}$ is *not* invertible.

We usually require the ARMA processes we use to be both stationary and invertible; all the zeros of both $\phi(B)$ and $\theta(B)$ must lie strictly outside the unit circle.

The models that SAS fits typically satisfy these conditions.