Identifying Transfer Functions for Step Interventions

Suppose

$$Y_t = v(B)X_t + N_t$$

where

$$v(B) = \frac{B^b \omega(B)}{\delta(B)}$$

and X_t is step function representing an intervention at time t_{event}:

$$X_t = \begin{cases} 1 & \text{if } t \geq t_{\text{event}} \\ 0 & \text{if } t < t_{\text{event}} \end{cases}.$$

We (tentatively) identify the form of the transfer function by comparing the pattern of the change in the series Y_t starting at time t_{event} with the pattern of

$$C_t = \frac{B^b \omega(B)}{\delta(B)} X_t$$

for various choices of $b \geq 0$,

$$\begin{aligned} \omega(B) =& \omega_0 - \omega_1 B - \omega_2 B^2 - \cdots - \omega_h B^h \text{, and} \\ \delta(B) =& 1 - \delta_1 B - \delta_2 B^2 - \cdots - \delta_r B^r \,. \end{aligned}$$

Choosing b

Choosing b (the delay) is easy. If $b = 0$, the series Y_t changes immediately at time t_{event}. If $b > 0$, the series does not change until time $t_{\text{event}} + b$. Frequently, $b = 0$.

The Case $r = 0$

If the series Y_t was stationary before the intervention and reaches a new permanent mean level a small number of time steps after the intervention, then we don't need a denominator; we can take $r = 0$. For particular values b and h, the series Y_t begins to change at time $t_{\text{event}} + b$ and reaches its new permanent (mean) level at time $t_{\text{event}} + b + h$.

If the series Y_t was non-stationary before the intervention, the situation is similar but more difficult to describe. At time $t_{\text{event}} + b + h$, the change in the series reaches it final value.

More precisely, if $b = 0$ and $r = 0$ so that

$$v(B) = \omega(B) = \omega_0 - \omega_1 B - \omega_2 B^2 - \cdots - \omega_h B^h,$$

then $Y_t = C_t + N_t$ where

$$C_t = \begin{cases} 0 & \text{for } t < t_{\text{event}} \\ \omega_0 & \text{for } t = t_{\text{event}} \\ \omega_0 - \omega_1 & \text{for } t = t_{\text{event}} + 1 \\ \omega_0 - \omega_1 - \omega_2 & \text{for } t = t_{\text{event}} + 2 \\ \vdots & \vdots \\ \omega_0 - \omega_1 - \cdots - \omega_{h-1} & \text{for } t = t_{\text{event}} + h - 1 \\ \omega_0 - \omega_1 - \cdots - \omega_h & \text{for } t \geq t_{\text{event}} + h \end{cases}$$

The Case $r = 1$

If $r = 1$, the change C_t converges exponentially to a limiting value from some time point onward.

For $r = 1$ and given values of b and h, we have $C_t = 0$ for $t < t_{\text{event}} + b$ and C_t converges exponentially to its limiting value starting from time $t_{\text{event}} + b + h$ onward. The rate of the exponential convergence is determined by δ_1. What goes on from time $t_{\text{event}} + b$ to time $t_{\text{event}} + b + h$ is determined by the values $\omega_0, \ldots, \omega_h$ and δ_1; anything is possible.

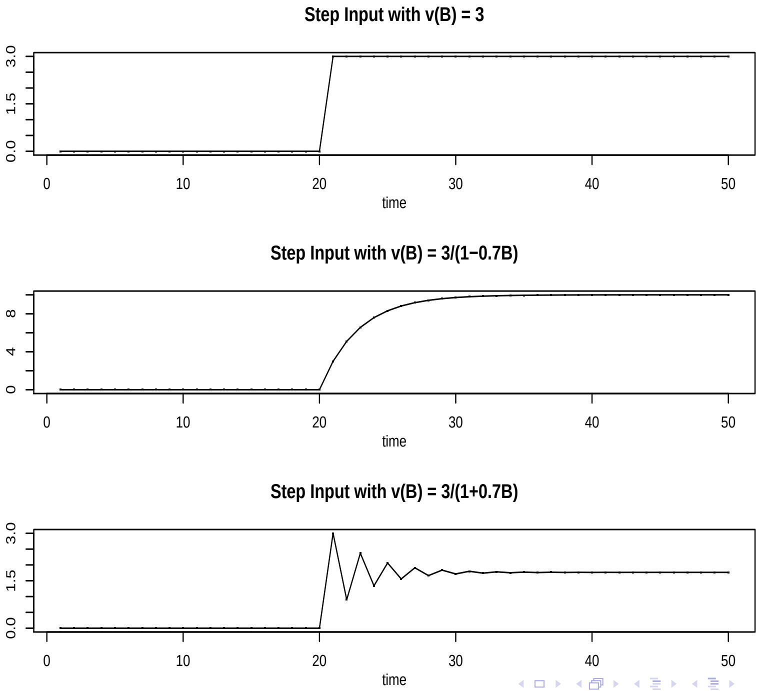
The simplest case is $r = 1$, $b = 0$, $h = 0$. In this case exponential convergence starts immediately at time t_{event}. In particular,

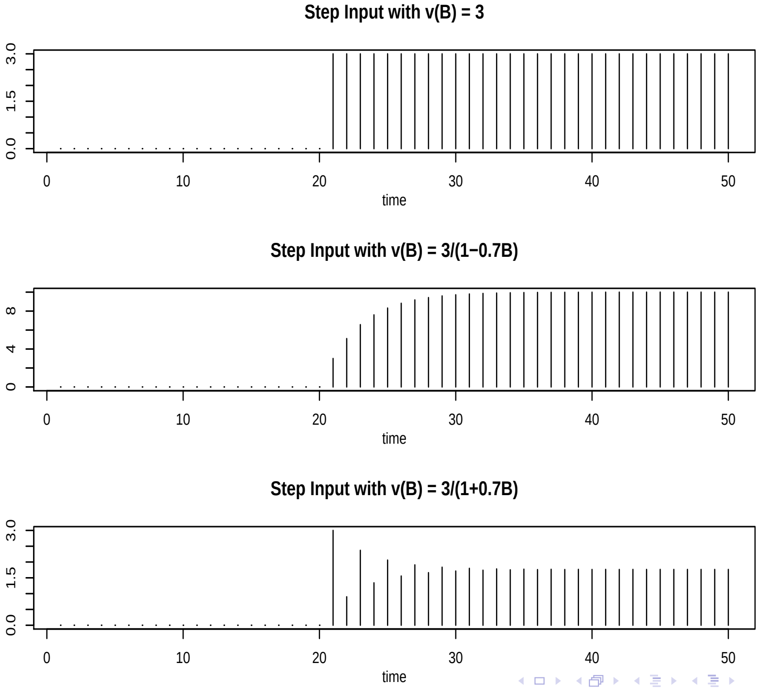
$$C_t = \begin{cases} \omega_0 & \text{for } t = t_{\text{event}} \\ \omega_0(1+\delta_1) & \text{for } t = t_{\text{event}} + 1 \\ \omega_0(1+\delta_1+\delta_1^2) & \text{for } t = t_{\text{event}} + 2 \\ \text{etc.} \end{cases}$$

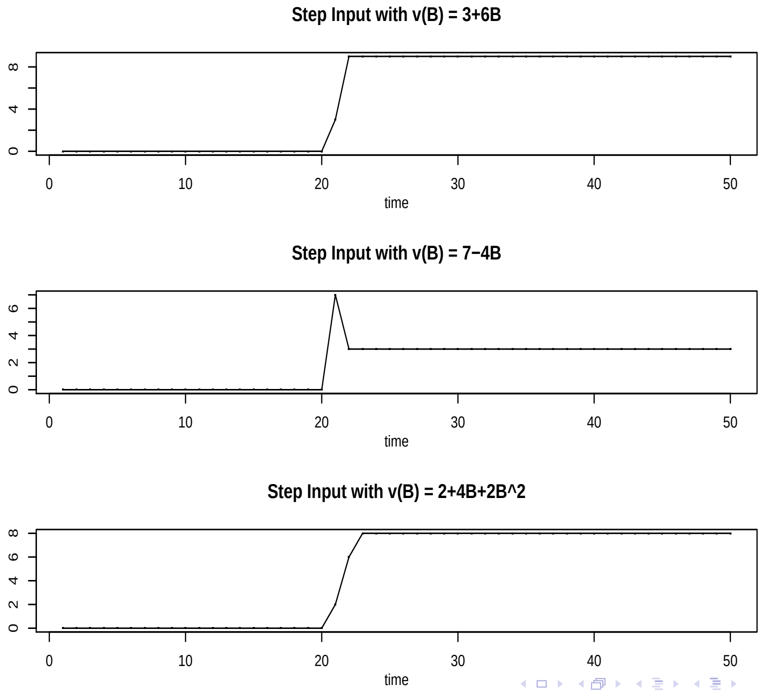
The Case $r = 2$

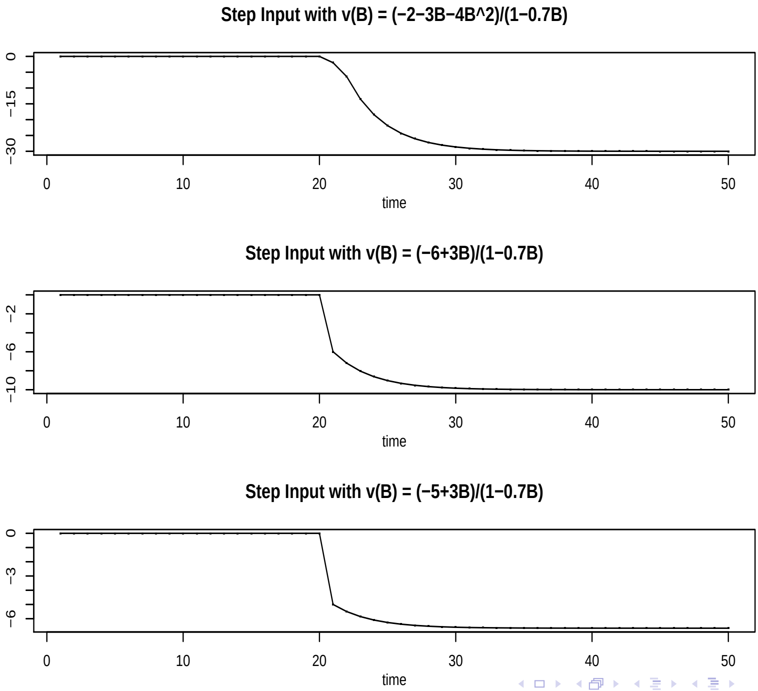
With $r = 2$ one can get a greater variety of behaviors, including sinusoidally oscillating exponential convergence.

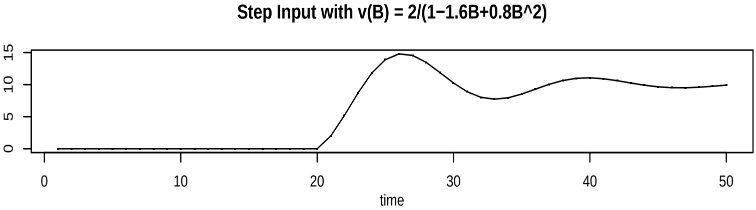
The following pages give plots of $C_t = v(B)X_t$ where X_t is a step function with $t_{\text{event}} = 21$.

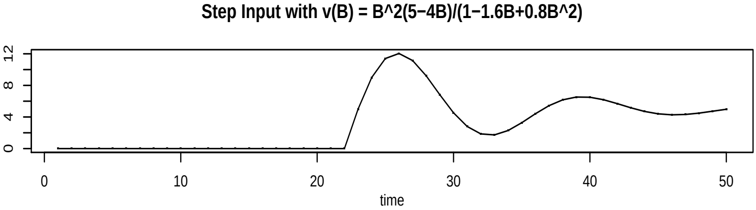

Step Input with v(B) = 3
3.0
1.5
0.0
0
10
20
30
40
50
time
Step Input with v(B) = 3/(1−0.7B)
8
4
0
0
10
20
30
40
50
time
Step Input with v(B) = 3/(1+0.7B)
3.0
1.5
0.0
0
10
20
30
40
50
time

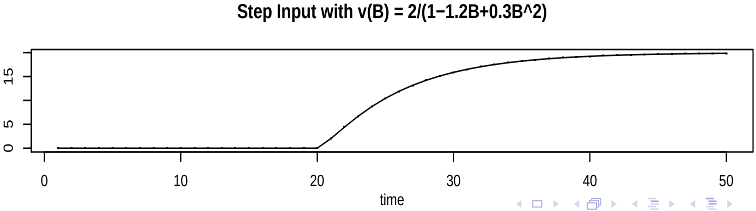

Step Input with v(B) = 3
0.0
1.5
3.0
0
10
20
30
40
50
time
Step Input with v(B) = 3/(1−0.7B)
0
4
8
0
10
20
30
40
50
time
Step Input with v(B) = 3/(1+0.7B)
0.0
1.5
3.0
0
10
20
30
40
50
time


Step Input with v(B) = 3+6B
Step Input with v(B) = 7−4B
Step Input with v(B) = 2+4B+2B^2
time
0
10
20
30
40
50
0
2
4
6
8


Step Input with v(B) = (−2−3B−4B^2)/(1−0.7B)
0
−15
−30
0
10
20
30
40
50
time
Step Input with v(B) = (−6+3B)/(1−0.7B)
−2
−6
−10
0
10
20
30
40
50
time
Step Input with v(B) = (−5+3B)/(1−0.7B)
0
−3
−6
0
10
20
30
40
50
time


Step Input with v(B) = 2/(1−1.6B+0.8B^2)
0
5
10
15
0
10
20
30
40
50
time


Step Input with v(B) = B^2(5−4B)/(1−1.6B+0.8B^2)
0
4
8
12
0
10
20
30
40
50
time


Step Input with v(B) = 2/(1−1.2B+0.3B^2)
0
5
15
0
10
20
30
40
50
time