$\ln(S_t) = X_t + \varepsilon_t$

S_t: Spot Price

X_t: short term deviation

ε_t: equilibrium price level

① (X_t) Ornstein-Uhlenbeck process

$dX_t = -k(X_t - 0)dt + \sigma_x dz_x$

k: mean reversion coefficient

σ_x: volatility of deviations

dz_x: correlated increment of standard Brownian Motion

② (ε_t) Brownian Motion Process

$d\varepsilon_t = \mu_\varepsilon dt + \sigma_\varepsilon dz_\varepsilon$

μ_ε: avg equilibrium growth over t

σ_ε: volatility of equilibrium level

dz_ε: correlated increment of standard Brownian motion

$$\rho_{x\varepsilon} = corr(dz_x, dz_\varepsilon) = \frac{cov(dz_x, dz_\varepsilon)}{\sqrt{V(z_x)V(z_\varepsilon)}} = \frac{E[dz_x \cdot dz_\varepsilon] - E[\overset{0}{dz_x}]\cdot E[\overset{0}{dz_\varepsilon}]}{\sqrt{dt\cdot dt}} \Rightarrow E[dz_x dz_\varepsilon] = \rho_{x\varepsilon}dt$$

$X_t = \emptyset X_{t-1} + n_{xt}$

$\varepsilon_t = \mu_\varepsilon \Delta t + \varepsilon_{t-1} + n_{\varepsilon t}$

$\emptyset = 1 - k\Delta t$

$$\xrightarrow{\text{matrix notation}} \begin{bmatrix} x_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_\varepsilon \Delta t \end{bmatrix} + \begin{bmatrix} \emptyset & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} n_{xt} \\ n_{\varepsilon t} \end{bmatrix}$$

$$X_t = c + Q \quad X_{t-1} + n_t$$

$$n_t \sim N(\mu, w) \; ; \; \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} \sigma_x^2 \Delta t & \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon \Delta t \\ \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon \Delta t & \sigma_\varepsilon^2 \Delta t \end{bmatrix}$$

N-step ahead mean vector

$m_0 = X_0 = \begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix}$, $m_t = E(X_t) = c + Q\,E(X_{t-1}) + E(n_t) = c + Q\,m_{t-1}$

$m_1 = c + Q\,m_0$

$m_2 = c + Q\,m_1 = c + Q[c + Q m_0] = c + Qc + Q^2 m_0 = c + c + Q^2 m_0 = 2c + Q^2 m_0$ — $\left[\text{note: } Qc = \begin{bmatrix} \emptyset & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \mu_\varepsilon\Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_\varepsilon \Delta t \end{bmatrix} = c\right.$

$\vdots$

$$m_n = nc + Q^n m_0 = n\begin{bmatrix} 0 \\ \mu_\varepsilon \Delta t \end{bmatrix} + \begin{bmatrix} \emptyset^n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix} = \begin{bmatrix} 0 \\ n\mu_\varepsilon\Delta t \end{bmatrix} + \begin{bmatrix} \emptyset^n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix}$$

— $\left[\text{note } Q^2 = Q\cdot Q = \begin{bmatrix} \emptyset & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \emptyset & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \emptyset^2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow Q^n = \begin{bmatrix} \emptyset^n & 0 \\ 0 & 1 \end{bmatrix}\right.$

$$= \begin{bmatrix} 0 \\ n\mu_\varepsilon\Delta t \end{bmatrix} + \begin{bmatrix} \emptyset^n x_0 \\ \varepsilon_0 \end{bmatrix} = \begin{bmatrix} \emptyset^n x_0 \\ \varepsilon_0 + n\mu_\varepsilon\Delta t \end{bmatrix}$$

N-step ahead variance vector

$V(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $V_t = Q V_t Q' + V(n_t) = Q V_t Q' + W$

$V_1 = Q\overset{0}{V_0}Q' + W = W$

$V_2 = Q V_1 Q' + W = QWQ' + W$

$V_3 = Q V_2 Q' + W = Q[QWQ' + W]Q' + W = [Q^2WQ' + QW]Q' + W$

$\quad = Q^2 W Q'^2 + QWQ' + W$

$\vdots$

$$V_n = Q^{n-1}WQ'^{n-1} + \dots QWQ' + W = \sum_{k=0}^{n-1} Q^k W Q'^k, \quad W = Q^0 W Q^0$$

$$\bullet\; Q^k W Q'^k = \begin{bmatrix} \emptyset^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_x^2\Delta t & \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t \\ \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t & \sigma_\varepsilon^2\Delta t \end{bmatrix}\begin{bmatrix} \emptyset^k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \emptyset^k\sigma_x^2\Delta t & \emptyset^k\rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t \\ \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t & \sigma_\varepsilon^2\Delta t \end{bmatrix}\begin{bmatrix} \emptyset^k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \emptyset^{2k}\sigma_x^2\Delta t & \emptyset^k\rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t \\ \emptyset^k\rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t & \sigma_\varepsilon^2\Delta t \end{bmatrix}$$

Note: $Q = \begin{bmatrix} \emptyset^k & 0 \\ 0 & 1 \end{bmatrix} = Q'$

$$\Rightarrow V_n = \begin{bmatrix} \sigma_x^2\Delta t \sum_{i=0}^{n-1}\emptyset^{2k} & \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t\sum_{i=0}^{n-1}\emptyset^k \\ \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\Delta t\sum_{i=0}^{n-1}\emptyset^k & n\sigma_\varepsilon^2\Delta t \end{bmatrix}$$

From Geometric Series

$$\sum_{i=0}^{n-1}\phi^i = \frac{1-\phi^{n-1}}{1-\phi}, \quad \sum_{i=0}^{n-1}\phi^{2i} = \frac{1-\phi^{2(n-1)}}{1-\phi^2}$$

• $\phi = 1-k\Delta t \Rightarrow \phi^n = (1-k(\frac{t}{n}))^n = (1-\frac{kt}{n})^n$

• $\Delta t = \frac{t}{n}$

$$\lim_{n\to\infty}\phi^n = \lim_{n\to\infty}(1-\tfrac{kt}{n})^n = e^{-kt}$$

$$\lim_{n\to\infty}\phi^{2n} = \lim_{n\to\infty}(\phi^n)^2 = e^{-2kt}$$

Note: $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

$$\lim_{n\to\infty}\Delta t\sum_{i=0}^{n-1}\phi^k = \frac{1-\phi^{n-1}}{1-\phi}\Delta t = \frac{1-\phi^{n-1}}{\frac{1-\phi}{\Delta t}} = \frac{1-e^{kt}}{k}$$

$$\lim_{n\to\infty}\Delta t\sum_{i=0}^{n-1}\phi^{2k} = \frac{1-\phi^{2(n-1)}}{1-\phi^2}\Delta t = \frac{1-\phi^{2(n-1)}}{\frac{1-\phi^2}{\Delta t}} = \frac{1-e^{-2kt}}{2k}$$

Note:

$$k = \frac{1-\phi}{\Delta t}$$

$$\phi^2 = (1-k\Delta t)^2 = 1-2k\Delta t + \overset{\to 0}{k^2(\Delta t)^2} \Rightarrow 2k\Delta t = 1-\phi^2$$

$$\Rightarrow 2k = \frac{1-\phi^2}{2K}$$

3a) long-run mean vector

$$\lim_{n\to\infty} m_n = E\begin{bmatrix}X_t\\ \xi_t\end{bmatrix} = \begin{bmatrix}\phi^n X_0\\ \xi_0+n\mu_\xi\Delta t\end{bmatrix} = \begin{bmatrix}e^{-kt}X_0\\ \xi_0+n\mu_\xi\frac{t}{n}\end{bmatrix} = \begin{bmatrix}e^{-kt}X_0\\ \xi_0+\mu_\xi t\end{bmatrix}$$

3b) long-run covariance vector

$$\lim_{n\to\infty} V_n = cov\begin{bmatrix}X_t\\ \xi_t\end{bmatrix} = \begin{bmatrix}\sigma_x^2\Delta t\sum_{i=0}^{n-1}\phi^{2k} & \rho_{x\xi}\sigma_x\sigma_\xi\Delta t\sum_{i=0}^{n-1}\phi^k\\ \rho_{x\xi}\sigma_x\sigma_\xi\Delta t\sum_{i=0}^{n-1}\phi^k & n\sigma_\xi^2\Delta t\end{bmatrix} = \begin{bmatrix}\frac{(1-e^{-2kt})}{2k}\sigma_x^2 & \frac{(1-e^{-kt})}{k}\rho_{x\xi}\sigma_x\sigma_\xi\\ \frac{(1-e^{-kt})}{k}\rho_{x\xi}\sigma_x\sigma_\xi & n\sigma_\xi^2(\frac{t}{n})\end{bmatrix}$$

$$= \begin{bmatrix}(1-e^{-2kt})\frac{\sigma_x^2}{2k} & (1-e^{-kt})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k}\\ (1-e^{-kt})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k} & \sigma_\xi^2 t\end{bmatrix}$$

4a) $E[\ln(S_t)] = E[X_t+\xi_t] = E[X_t]+E[\xi_t] = e^{-kt}X_0+\xi_0+\mu_\xi t$

4b) $V[\ln(S_t)] = V[X_t+\xi_t] = V(X_t)+V(\xi_t)+2\,cov(X_t,\xi_t)$

$$= (1-e^{-2kt})\frac{\sigma_x^2}{2k}+\sigma_\xi^2 t+2(1-e^{-kt})\rho_{x\xi}\frac{\sigma_x\sigma_\xi}{k}$$

• General fact: $Y\sim N(\mu,\sigma)$, $X=e^Y\sim \log N[e^{\mu+\frac{\sigma^2}{2}},(e^{\sigma^2}-1)(e^{2\mu+\sigma^2})]$

$\ln(S_t)$ assumed to be normally distributed

$$S_t = e^{\ln(S_t)}\sim \log Norm(e^{E[\ln(S_t)]+\frac{1}{2}V[\ln(S_t)]}, \square)$$

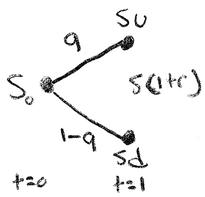
5) $E[S_t] = e^{E[\ln(S_t)]+\frac{1}{2}V[\ln(S_t)]}$ or $\ln[E(S_t)] = E[\ln(S_t)]+\frac{1}{2}V[\ln(S_t)]$

$$= e^{-kt}X_0+\xi_0+\mu_\xi t+\frac{1}{2}\left[(1-e^{-2kt})\frac{\sigma_x^2}{2k}+\sigma_\xi^2 t+2(1-e^{-kt})\rho_{x\xi}\frac{\sigma_x\sigma_\xi}{k}\right]$$

6) $\lim_{t\to\infty}\ln[E(S_t)] = 0+\xi_0+\mu_\xi t+\frac{1}{4k}\sigma_x^2+\frac{1}{2}\sigma_\xi^2 t+\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k}$

$$= \left(\xi_0+\frac{\sigma_x^2}{4k}+\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k}\right)+\left(\mu_\xi+\frac{1}{2}\sigma_\xi^2\right)t$$

Risk Neutral Valuation



$$E(S)_{t=1} = q(Su) + (1-q)(Sd) \Rightarrow S = \frac{E(S)}{1+k} = \frac{q(Su)+(1-q)Sd}{1+k}$$

$$\Rightarrow Su > S(1+r) > Sd \Rightarrow u > 1+r > d \text{ , find weight } p \text{ s.t. } p(u)+(1-p)d = 1+r$$

$$\Rightarrow p = \frac{1+r-d}{u-d} \qquad S[p(u)+(1-p)d] = S(1+r) \Rightarrow S = \frac{p(Su)+(1-p)Sd}{1+r}$$

$$E(R) = E\left(\frac{S_{t=1}-S_0}{S_0}\right) = \frac{E(S_{t=1})-S_0}{S_0} = \frac{q(S_0u)+(1-q)Sd - S_0}{S_0} = qu+(1-q)d-1$$

Risk Premium: $\alpha = \dfrac{E(R)-r}{\sigma_r}$

r_i:	u	d
$p(r_i)$	q	$(1-q)$

$$\sigma_r^2 = \sum_{i=1}^{2}(r_i - E(r_i))^2 p(r_i)$$
$$= [u-(qu+(1-q)d)]^2 q + [d-(qu+(1-q)d)]^2(1-q)$$
$$= [(1-q)^2(u-d)^2]q + [q^2(d-u)^2](1-q)$$
$$= q(1-q)[(1-q)(u-d)^2 + q(d-u)^2] = q(1-q)(u-d)^2$$

$$\Rightarrow \alpha = \frac{qu+(1-q)d-1-r}{\sqrt{q(1-q)(u-d)^2}} = \frac{qu+d-qd-(1+r)}{\sqrt{q(1-q)}\,(u-d)}$$

$$\Rightarrow \alpha\sqrt{q(1-q)} = \frac{q(u-d)-(1+r-d)}{u-d} = q - \frac{(1+r-d)}{u-d} = q-p$$

$$\Rightarrow \alpha\sqrt{q(1-p)} = q-p \quad \text{so} \quad p = q - \alpha\sqrt{q(1-p)}$$

Model: $\dfrac{dS_t}{S_t} = R_t = E(R)dt + \sigma\,\varepsilon_t\sqrt{dt} \qquad \varepsilon_t \sim N(0,1)$

$E(R) = qu + (1-q)d - 1$ return under actual probability q

$E^*(R) = (q-\alpha\sqrt{q(1-q)}) + (1-[q-\alpha\sqrt{q(1-q)}])d - 1$ return under risk neutral probability p

$= qu + (1-q)d - 1 - \alpha\sqrt{q(1-q)}\,(u-d)$

$\Rightarrow E^*(R) = E(R) - \lambda$

$\lambda = \alpha\sqrt{q(1-q)}\,(u-d)$

$$R_t^* = E^*(R)dt + \sigma\,\varepsilon_t\sqrt{dt}$$
$$= (E(R)-\lambda)dt + \sigma\,\varepsilon_t\sqrt{dt}$$

R_t^*: risk neutral process

Risk neutral Process

$\therefore \quad dX_t = -kX_t\,dt + \sigma_x\,dz_x \longrightarrow$ (7a) $dX_t = (-kX_t - \lambda_x)dt + \sigma_x\,dz_x^*$

$d\xi_t = \mu_\xi\,dt + \sigma_\xi\,dz_\xi \longrightarrow$ (7b) $d\xi_t = \underbrace{(\mu_\xi - \lambda_\xi)}_{\mu_\xi^*}dt + \sigma_\xi\,dz_\xi^*$

$$dX_t = -k\left[X_t - \left(-\frac{\lambda_x}{k}\right)\right]dt + \sigma_x\,dz_x$$

- Ornstein-Uhlenbeck process is now mean reverting toward $-\frac{\lambda_x}{k}$

$$d\xi_t = \mu_\xi^*\,dt + \sigma_\xi\,dz_\xi$$

- Brownian motion now has drift μ_ξ^*

Mean vectors and covariance structure changes

$$E\begin{bmatrix} X_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} e^{-kt}X_0 \\ \xi_0 + \mu_\xi t \end{bmatrix} \longrightarrow E^*\begin{bmatrix} X_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \xi_0 + \mu_\xi^* t \end{bmatrix}$$

*must calculate since Mean reverting parameter effects process

just replace μ_ξ by $\mu_\xi^* = \mu_\xi - \lambda_\xi$ since constant through process

$$\mathrm{Cov}\begin{bmatrix} X_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} (1-e^{-2kt})\frac{\sigma_x^2}{2k} & (1-e^{-kt})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k} \\ (1-e^{-kt})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k} & \sigma_\xi^2 t \end{bmatrix} \longrightarrow \mathrm{Cov}^*\begin{bmatrix} X_t \\ \xi_t \end{bmatrix} = \mathrm{Cov}\begin{bmatrix} X_t \\ \xi_t \end{bmatrix}$$

since doesn't depend on drift or mean reverting parameters μ_ξ^*, $-\frac{\lambda_x}{k}$

Ornstein-Uhlenbeck Process

$dx_t = -k(x_t - \theta)dt + \sigma dw_t$

θ: long term mean of process

Ignoring random fluctuation from dw_t

$dx_t = k(\theta - x)dt \Rightarrow \frac{dx_t}{x_t - \theta} = -k\,dt \Rightarrow \int_{x_0}^{x_t} \frac{dx_t}{(x_t-\theta)} = -k\int_{t=0}^{t} dt$

$\Rightarrow \ln(x_t - \theta)\Big|_{x_0}^{x_t} = -kt \Rightarrow \ln(x_t-\theta) - \ln(x_0 - \theta) = -kt \Rightarrow \ln\left(\frac{x_t - \theta}{x_0-\theta}\right) = -kt$

$\Rightarrow \frac{x_t-\theta}{x_0-\theta} = e^{-kt} \Rightarrow x_t = \theta + (x_0-\theta)e^{-kt}$ as $t \to \infty$ $x_t \to \theta$ at rate k

$(x_t-\theta) = (x_0-\theta)e^{-kt}$: suggests $y_t = e^{-kt} z_t \Leftrightarrow z_t = e^{kt} y_t$

let $y_t = x_t - \theta \Rightarrow dy_t = -k y_t dt + \sigma dw_t \Rightarrow y_t \to 0$ at rate k,

$\frac{dz_t}{dt} = ke^{kt}y_t + e^{kt}\frac{dy_t}{dt} \Rightarrow dz_t = ke^{kt}y_t dt + e^{kt}dy_t = ke^{kt}y_t dt + e^{kt}(-ky_t dt + \sigma dw_t) = \sigma e^{kt} dw_t$

$\therefore dz_t = \sigma e^{kt} dw_t \to \int_{s=0}^{t} dz_t = \int_{s=0}^{t} \sigma e^{kt} dw_t = z_t - z_0 = \int_0^t \sigma e^{kt} dw_t \Rightarrow z_t = z_0 + \sigma \int_0^t e^{ku} dw_u$, $z_t = e^{kt} y_t$

$\Rightarrow e^{kt} y_t = e^{k(0)} y_0 + \sigma \int_0^t e^{ku} dw_u \Rightarrow y_t = e^{-kt} y_0 + \sigma e^{-kt} \int_0^t e^{ku} dw_u$

$x_t = y_t + \theta \Rightarrow x_t = \theta + e^{-kt}(x_0 - \theta) + \sigma e^{-kt} \int_0^t e^{ku} dw_u$

Note: $\int_a^b x_t dw_t \stackrel{\text{defn}}{=} \sum_{i=0}^{k-1} x_{t_i}(w_{t_{i+1}} - w_{t_i})$

$w_t \sim N(0, t)$

$E(x_t) = \theta + e^{-kt}(x_0 - \theta) + \sigma e^{-kt} E\left[\int_0^t e^{ku} dw_u\right]$

$E\left[\int_0^t \underbrace{e^{ku}}_{x_t} dw_u\right] = \sum_{i=0}^{k-1} \underbrace{x_{t_i}}_{\text{constant on interval}} \overbrace{E[w_{t_{i+1}} - w_{t_i}]}^{0} = 0$

$\Rightarrow E(x_t) = \theta + e^{-kt}(x_0 - \theta) = \theta + e^{-kt}x_0 - \theta e^{-kt} = e^{-kt}x_0 + \theta(1 - e^{-kt})$

$\theta = -\frac{\Lambda_x}{k}$ in our problem

$\therefore E(x_t) = e^{-kt}x_0 - (1 - e^{-kt})\frac{\Lambda_x}{k}$

$\therefore E^*\begin{bmatrix} x_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} e^{-kt}x_0 - (1-e^{-kt})\frac{\Lambda_x}{k} \\ \xi_0 + \mu_\xi^* t \end{bmatrix}$

(8a) $\Rightarrow E^*[\ln(S_t)] = E^*[x_t + \xi_t] = e^{-kt}x_0 - (1 - e^{-kt})\frac{\Lambda_x}{k} + \xi_0 + \mu_\xi^* t = e^{-kt}x_0 + \xi_0 - (1 - e^{-kt})\frac{\Lambda_x}{k} + \mu_\xi^* t$

(8b) $\text{var}^*[\ln(S_t)] = \text{var}^*[x_t + \xi_t] = \text{var}^*[x_t] + \text{var}^*[\xi_t] + 2\text{cov}^*[x_t, \xi_t] = \text{Var}[x_t] + \text{Var}[\xi_t] + 2\text{cov}[x_t, \xi_t]$

$= \text{Var}[\ln(S_t)]$

Futures price is risk neutral expected spot price $\left[\frac{E^*(S)}{p(S_u) + (1-p)(S_d)} = S(1+r) \stackrel{\text{defn}}{=} F\right]$

(9) $\ln(F_{T,0}) = \ln(E^*(S_T)) = E^*[\ln(S_t)] + \frac{1}{2}V^*[\ln(S_t)] = e^{-kt}x_0 + \xi_0 - (1 - e^{-kt})\frac{\Lambda_x}{k} + \mu_\xi^* t + \frac{1}{2}\left[(1 - e^{-2kt})\frac{\sigma_x^2}{2k} + \sigma_\xi^2 t + 2(1 - e^{-kt})\rho_{x\xi}\frac{\sigma_x \sigma_\xi}{k}\right]$

$= e^{-kt}x_0 + \xi_0 \underbrace{+ \mu_\xi^* t - (1 - e^{-kt})\frac{\Lambda_x}{k} + \frac{1}{2}\left((1 - e^{-2kt})\frac{\sigma_x^2}{2k} + \sigma_\xi^2 t + 2(1 - e^{-kt})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{k}\right)}_{A(T)}$

$= e^{-kt}x_0 + \xi_0 + A(t)$