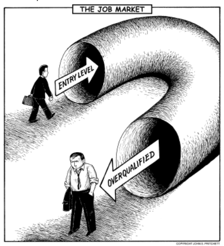


The Greek Letters

The Black-Scholes Formulas

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Example

- **A bank has sold for $300,000 a European call option on 100,000 shares of a nondividend paying stock**
- **S_0 = 49, X = 50, r = 5%, σ = 20%, T = 20 weeks**

  The Black-Scholes value of the option is $240,000
- **How does the bank hedge its risk?**

Naked & Covered Positions

Naked position

Take no action

Covered position

Buy 100,000 shares today

Both strategies leave the bank exposed to significant risk

Stop-Loss Strategy

This involves:

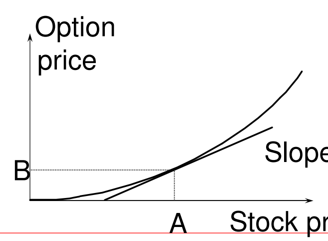
- Buying 100,000 shares as soon as price reaches $50
- Selling 100,000 shares as soon as price falls below $50

This deceptively simple hedging strategy does not work well

Why?

Delta

- Delta (Δ) is the rate of change of the option price with respect to the underlying



Slope = $\Delta = \frac{\partial C}{\partial S}$

Delta Hedging

- **This involves maintaining a delta neutral portfolio**
- **The delta of a European call on a stock paying dividends at rate q is**

$$N(d_1)e^{-qT}$$

- **The delta of a European put is**

$$e^{-qT}[N(d_1)-1]$$

What is the delta of a forward contract?

Delta Hedging continued

- **The hedge position must be frequently rebalanced**
- **Delta hedging a written option involves a "buy high, sell low" trading rule**

Option Closes in the Money: Cost to the writer=$258,110

Week	Stock Price	Delta	Shares Purchased	Cost (000)	Cumulative Costs (000)	Interests Cost
0	**49.00**	**0.522**	**52,160**	**2,555.86**	**2,555.86**	**2.46**
1	**48.13**	**0.458**	**(6,325)**	**(304.41)**	**2,253.91**	**2.17**
2	**47.38**	**0.400**	**(5,798)**	**(274.67)**	**1,981.41**	**1.91**
3	**50.25**	**0.596**	**19,591**	**984.44**	**2,967.76**	**2.85**
4	**51.75**	**0.693**	**9,667**	**500.27**	**3,470.88**	**3.34**
5	**53.36**	**0.786**	**9,352**	**499.06**	**3,973.27**	**3.82**
6	**53.64**	**0.805**	**1,819**	**97.55**	**4,074.64**	**3.92**
7	**53.78**	**0.817**	**1,243**	**66.86**	**4,145.42**	**3.99**
8	**52.63**	**0.759**	**(5,850)**	**(307.87)**	**3,841.53**	**3.69**
9	**52.58**	**0.761**	**210**	**11.02**	**3,856.25**	**3.71**
10	**52.20**	**0.741**	**(2,002)**	**(104.50)**	**3,755.45**	**3.61**
11	**53.50**	**0.831**	**9,048**	**484.06**	**4,243.12**	**4.08**
12	**53.78**	**0.857**	**2,597**	**139.68**	**4,386.87**	**4.22**
13	**50.38**	**0.591**	**(26,607)**	**(1,340.32)**	**3,050.77**	**2.93**
14	**52.13**	**0.768**	**17,673**	**921.20**	**3,974.90**	**3.82**
15	**51.88**	**0.759**	**(859)**	**(44.56)**	**3,934.16**	**3.78**
16	**52.88**	**0.866**	**10,659**	**563.66**	**4,501.61**	**4.33**
17	**54.88**	**0.978**	**11,257**	**617.71**	**5,123.65**	**4.93**
18	**54.63**	**0.990**	**1,159**	**63.34**	**5,191.91**	**4.99**
19	**55.83**	**1.000**	**1,003**	**55.99**	**5,252.89**	**5.05**
20	**57.25**	**1.000**	**3**	**0.17**	**5,258.11**	

Option Closes out of the Money: Cost to the writer=$236,440

Week	Stock Price	Delta	Shares Purchased	Cost (000)	Cumulative Costs (000)	Interests Cost
0	49.00	0.522	52,160	2,555.86	2,555.86	2.46
1	48.85	0.507	(1,416)	(69.18)	2,489.14	2.39
2	47.96	0.441	(6,643)	(318.60)	2,172.93	2.09
3	46.03	0.301	(14,038)	(646.21)	1,528.81	1.47
4	48.16	0.443	14,189	683.26	2,213.54	2.13
5	46.42	0.307	(13,542)	(628.60)	1,587.07	1.53
6	47.14	0.350	4,247	200.21	1,788.80	1.72
7	45.99	0.254	(9,560)	(439.62)	1,350.90	1.30
8	46.34	0.267	1,275	59.07	1,411.27	1.36
9	46.27	0.247	(1,940)	(89.74)	1,322.89	1.27
10	45.45	0.175	(7,234)	(328.76)	995.40	0.96
11	44.39	0.100	(7,541)	(334.78)	661.58	0.64
12	44.02	0.069	(3,094)	(136.22)	525.99	0.51
13	45.14	0.103	3,429	154.77	681.27	0.66
14	43.57	0.028	(7,462)	(325.15)	356.77	0.34
15	43.79	0.021	(717)	(31.37)	325.74	0.31
16	43.49	0.008	(1,333)	(57.95)	268.10	0.26
17	41.81	0.000	(766)	(32.02)	236.34	0.23
18	42.42	0.000	(12)	(0.50)	236.07	0.23
19	41.61	0.000	(2)	(0.08)	236.22	0.23
20	41.85	0.000	(0)	(0.00)	236.44	

Using Futures for Delta Hedging

- **The delta of a futures contract is $e^{(r-q)T}$ times the delta of a spot contract**
- **The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract**
- **For foreign exchange the position is $e^{-r_f T}$**

Example:

A bank has written a 6-month option to sell 1,000,000 pounds at an exchange rate of 1.6. The current exchange rate is 1.62, the RFR in the UK is 13%, in the U.S. it is 10%, and the volatility of sterling is 15%.

Thus $S0$ = 1.62, X = 1.60, r = 10%, rp = 13%, s = 15%, T=.5

What is the option's delta?

How do we interpret the results?

How can we use a futures contract instead?

Let us examine the Bank Risk

- The bank wrote a put option for £1,000,000 at 1.6$/£ in 6 months
  - **What does this imply?**
  - **The counterparty can exchange £1,000,000 for $1,600,000 in 6 months time if they choose to do so.**
  - **When does the counterparty exercise?**
  - **Only when the exchange rate is less than 1.6$/£**
- Now, How does the bank hedge the risk?
  - **Short sterling**
  - **How much?**
  - **£457,794**
- How does that neutralize the risk.
  - **The banks current delta position is short -1,000,000 X -.457794=457,794**
  - **Short sterling induces a -457,794 delta hedge position**
  - **This results in a delta-neutral portfolio.**

Why does shorting make sense?

- Let us examine exchange rate movements
  - **Assume the exchange rate fell to 1.59 $/£**
  - **The option is in the money and the counterpart could exercise the option for a net gain of $10,000**
  - **For the hedge to be effective, the short sterling hedged position must make money to cover potential losses from the written option.**
  - **Well, when the bank sold sterling forward, the bank received**
    - **£457,794 X 1.62$/£ =$741,626**
  - **If the exchange rate fell the short position increased in value from £457,794 to**
    - **$741,626 / 1.59$/£ = £466,431**
  - **Thus the hedge is successful**
  - **The same logic applies if the exchange rate increased, but this time the option would not be exercised and the short position would lose money**

Alternate hedge

- Let us use a futures contract instead.
  - **What is the position?**
  - **Take a short position in currency futures to delivery sterling in exchange for dollars**
  - **Let us assume a 9-month futures contract is available, the appropriate hedge ratio is**
    - **Exp(-(13%-10%)*.75)*(457,794)=£468,442**
  - **If each contract is worth £62,500, 7 contracts should be shorted**

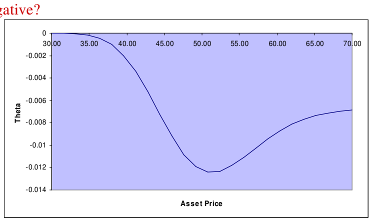
Theta

- **Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time**

Think of theta as the *time decay* of the portfolio

$$\Theta = \frac{S_0\, N'(d_1)\sigma}{2\sqrt{T}} - rX\, e^{-rT} N(d_2)$$

Why is Theta negative?

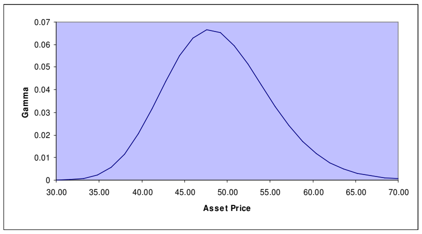


Gamma

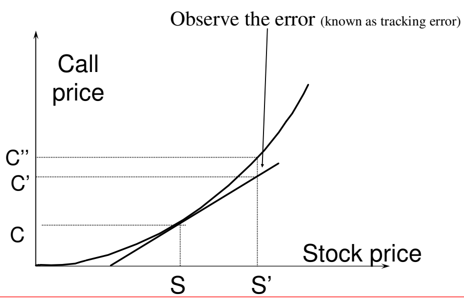
- **Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset**

Small Gamma results effects rebalancing in what way?

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$



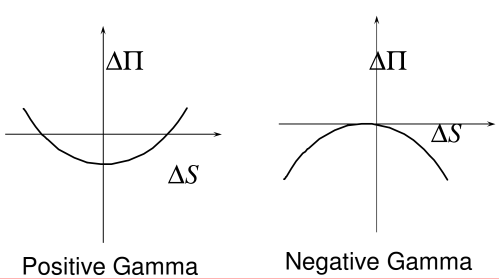
Gamma Addresses Delta Hedging Errors Caused By Curvature



Interpretation of Gamma

- **For a delta neutral portfolio,**

$\Delta\Pi$ » $\Theta \Delta t + \frac{1}{2}\Gamma \Delta S^2$



Relationship Among Delta, Gamma, and Theta

For a portfolio of derivatives on a stock paying a continuous dividend yield at rate q

$$\Theta + (r-q)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Interpretation and a question

- **Consider a 4 month put option on a stock index. Suppose the current value of the index is 305, the strike price is 300, the dividend yield is 3%, the RFR is 8%, and the volatility of the index is 25%**
- **The gamma is 0.00857**
- **Thus a 1 dollar move in the index results in a 0.0087 move in delta**
- **Suppose a delta-neutral portfolio has a gamma of -3000. The delta and gamma of a traded call option are .62 and 1.50 respectively. How do we make the portfolio gamma neutral? What happens to delta**

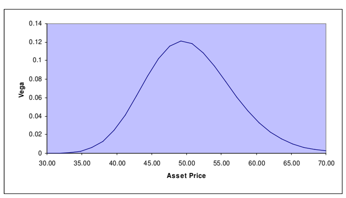
Vega

- **Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility**

$$\nu = S_0 \sqrt{T} N'(d_1) e^{-qT}$$

If vega is high, is the portfolio sensitive to changes in volatility?

As time increases, what happens to vega?



Managing Delta, Gamma, & Vega

- Delta, Δ, can be changed by taking a position in the underlying asset
- To adjust gamma, Γ, and vega, ν, it is necessary to take a position in an option or other derivative
- Vega neutrality is $-\upsilon/\upsilon_t$

Question

- Consider a portfolio that is delta neutral, with a gamma of -5000 and a vega of -8000. Suppose a traded option has a gamma of .5, a vega of 2, and a delta of .6.
- How do we make the portfolio vega neutral?
- What happens to delta and gamma?
- How to then make the portfolio vega and gamma neutral?

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate
- For currency options there are 2 rhos

$$rho = XTe^{-rT}\,\mathrm{N(d2)}$$
$$rho = -Te^{-r_f T}\,\mathrm{S}_0\,\mathrm{N(d1)}$$

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- As portfolio becomes larger hedging becomes less expensive

Hedging vs Creation of an Option Synthetically

- When we are hedging we take positions that offset Δ, Γ, ν, etc.
- This may be more attractive because of the size of the trade or lack of available strike prices
- When we create an option synthetically we take positions that match $\Delta, \Gamma, \& \nu$

Portfolio Insurance

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

Portfolio Insurance continued

- **As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased**
- **As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold**

Portfolio Insurance continued

The strategy did not work well on October 19, 1987...