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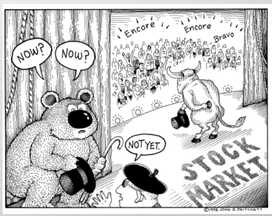
Estimating Volatilities and Correlations



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Standard Approach to Estimating Volatility

- **Define σ_n as the volatility per day between day n-1 and day n, as estimated at end of day n-1**
- **Define S_i as the value of market variable at end of day i**
- **Define $u_i = \ln(S_i/S_{i-1})$**

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

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Simplifications Usually Made

- **Define u_i as $(S_i - S_{i-1})/S_{i-1}$**
- **Assume that the mean value of u_i is zero**
- **Replace m-1 by m**

This gives

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

ARCH(m) Model

In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

EWMA Model

- **In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time**
- **This leads to**

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2$$

Attractions of EWMA

- **Relatively little data needs to be stored**
- **We need only remember the current estimate of the variance rate and the most recent observation on the market variable**
- **Tracks volatility changes**
- **RiskMetrics uses $\lambda = 0.94$ for daily volatility forecasting**

GARCH (1,1)

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

Example

- **Suppose**

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- **The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%**

Example continued

- **Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.**
- **The new variance rate is**

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

Maximum Likelihood Methods

- **In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring**

Example 1

- **We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, *p*, that it happens?**
- **The probability of the event happening on one particular trial and not on the others is**

$$p(1-p)^9$$

- **We maximize this to obtain a maximum likelihood estimate. Result: *p*=0.1**

Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize: $$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(\frac{-u_i^2}{2v}\right)\right]$$

Taking logarithms this is equivalent to maximizing:

$$\sum_{i=1}^{m}\left[-\ln(v)-\frac{u_i^2}{v}\right]$$

Result: $$v=\frac{1}{m}\sum_{i=1}^{m}u_i^2$$

Application to GARCH

We choose parameters that maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i}\right)$$

or

$$\sum_{i=1}^{m}\left[-\ln(v_i) - \frac{u_i^2}{v_i}\right]$$

Excel Application

- **Start with trial values of ω, α, and β**
- **Update variances**
- **Calculate** $\sum_{i=1}^{m}\left[-\ln(v_i) - \frac{u_i^2}{v_i}\right]$
- **Use solver to search for values of ω, α, and β that maximize this objective function**
- **Important note: set up spreadsheet so that you are searching for three numbers that are the same order of magnitude**

Variance Targeting

- **One way of implementing GARCH(1,1) that increases stability is by using variance targeting**
- **We set the long-run average volatility equal to the sample variance**
- **Only two other parameters then have to be estimated**

How Good is the Model?

- **The Ljung-Box statistic tests for autocorrelation**
- **We compare the autocorrelation of the u_i^2 with the autocorrelation of the u_i^2/σ_i^2**

Forecasting Future Volatility

A few lines of algebra shows that

$$E[\sigma^2_{n+k}] = V_L + (\alpha+\beta)^k(\sigma^2_n - V_L)$$

The variance rate for an option expiring on day *m* is

$$\frac{1}{m}\sum_{k=0}^{m-1} E\left[\sigma^2_{n+k}\right]$$

Forecasting Future Volatility continued

Define

$$a = \ln\frac{1}{\alpha+\beta}$$

The volatility per annum for a T-day option is

$$\sqrt{252\left(V_L + \frac{1-e^{-aT}}{aT}\left[V(0) - V_L\right]\right)}$$

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Volatility Term Structures

- **The GARCH (1,1) suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities**
- **Impact of 1% change in instantaneous volatility for Japanese yen example:**

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.84	0.61	0.46	0.27	0.06

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Correlations and Covariances

Define $x_i=(X_i-X_{i-1})/X_{i-1}$ and $y_i=(Y_i-Y_{i-1})/Y_{i-1}$
Also
$\sigma_{x,n}$: daily vol of X calculated on day n-1
$\sigma_{y,n}$: daily vol of Y calculated on day n-1
cov_n: covariance calculated on day n-1
The correlation is $\text{cov}_n/(\sigma_{u,n}\, \sigma_{v,n})$

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Updating Correlations

- **We can use similar models to those for volatilities**
- **Under EWMA**

$$\text{cov}_n = \lambda\, \text{cov}_{n-1} + (1-\lambda) x_{n-1} y_{n-1}$$

Positive Finite Definite Condition

A variance-covariance matrix, Ω, is internally consistent if the positive semi-definite condition

$$\mathbf{w}^{T}\Omega\,\mathbf{w} \geq 0$$

for all vectors w

Example

The variance covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent