Dynamic Optimization Examples

Example (Sequential Formulation): It is useful to have an example that has closed form solution. Assume that preferences are of the form $u(c) = \log c$. In this particular case, the optimization problem is

$$v_T(W_1) = \max \textstyle\sum_{t=1}^{T} \beta^{t-1} \log(c_t)$$
$$s.t. \qquad \textstyle\sum_{t=1}^{T} c_t = W_1,$$

Let λ be the Lagrange multiplier of the resource constraint, then we can write down the Lagrangian as

$$v_T(W_1) = \max_{\{\{c_t\}_{t=1}^{T}, \lambda\}} \textstyle\sum_{t=1}^{T} \beta^{t-1} \log(c_t) + \lambda[W_1 - \textstyle\sum_{t=1}^{T} c_t],$$

The Euler equation implies

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}$$

or

$$\gamma_c = \frac{c_{t+1}}{c_t} = \beta < 1$$

The growth rate of consumption is decreasing, that implies $c_1 > c_2 > .... > c_T$. In general, we can write consumption in period c_t as a function of the initial period c_1

$$c_t = \beta^{t-1} c_1,$$

Now, we can replace this value in the budget constraint to calculate the optimal consumption at c_1. Formally,

$$\textstyle\sum_{t=1}^{T} \beta^{t-1} c_1 = W_1,$$

or

$$c_1^* = \frac{1}{\sum_{t=1}^{T} \beta^{t-1}} W_1,$$

Now, we can calculate the optimal consumption value for any period t. Formally,

$$c_t^* = \frac{\beta^{t-1}}{\sum_{t=1}^{T} \beta^{t-1}} W_1$$

This expression can be used to calculate the optimal consumption sequence as a function of β and W_1, that is $\{c_t^*\}_{t=1}^{T}$. The optimal consumption sequence can be used to calculate the value function, that determines the optimal return function given an initial value W_1. That is

$$v_T(W_1) = \textstyle\sum_{t=1}^{T} \beta^{t-1} \log\left(\frac{\beta^{t-1}}{\sum_{t=1}^{T} \beta^{t-1}} W_1\right).$$

Rearranging terms,

$$v_T(W_1) = \textstyle\sum_{t=1}^{T} \beta^{t-1} \log\left(\frac{\beta^{t-1}}{\sum_{t=1}^{T} \beta^{t-1}}\right) + \textstyle\sum_{t=1}^{T} \beta^{t-1} \log(W_1)$$

where the value function is log-linear in W_1. We can rewrite this expression as

$$v_T(W_1) = A_T + B_T \log(W_1)$$

where $A_T = \sum_{t=1}^{T} \beta^{t-1} \log(\frac{\beta^{t-1}}{\sum_{t=1}^{T} \beta^{t-1}})$ and $B_T = \sum_{t=1}^{T} \beta^{t-1}$. The value function depends on the horizon of the problem. Since the utility function is concave, the utility function is increasing in t, that follows directly from the definition of concavity $u((1-\beta)c_1 + \beta c_2) < (1-\beta)u(c_1) + \beta u(c_2)$. We can see some specific cases:

1. $T = 1$: The implied value function is

$$v_1(W_1) = 1 \log(1) + \beta^0 \log(W_1) = 0 + 1 \log(W_1) = \log(W_1)$$

2. $T = 2$: In this case,

$$v_2(W_1) = \ln(\frac{1}{1+\beta}) + \beta \ln(\frac{\beta}{1+\beta}) + (1+\beta)\log(W_1),$$

3. $T = 3$: In this case,

$$v_3(W_1) = \log(\frac{1}{1+\beta+\beta^2}) + \beta \log(\frac{\beta}{1+\beta+\beta^2}) + \beta^2 \log(\frac{\beta^2}{1+\beta+\beta^2}) + (1+\beta+\beta^2)\log(W_1),$$

Example (Recursive Formulation): Next, we show how to construct the value function and the optimal decision rules recursively, when $u(c) = \ln c$. We begin with the last period and solve it backwards.

- $T = 1$,

$$v_1(W_1) = \max_{c_1} \ln(c_1)$$

$$s.to. \qquad W_2 = W_1 - c_1$$

  The optimal choice implies, $c_1^* = W_1$ and $W_2 = 0$. The implied value function is

$$v_1(W_1) = \ln(W_1)$$

- $T = 2$

$$v_2(W_1) = \max_{W_2}\{\ln(W_1 - W_2) + \beta v_1(W_2)\}$$

  given that we know the value function on the last period

$$v_2(W_1) = \max_{W_2}\{\ln(W_1 - W_2) + \beta \ln(W_2)\}$$

  the FOC imply

$$\frac{1}{W_1 - W_2} = \frac{\beta}{W_2},$$

  or

$$W_2^* = \frac{\beta}{1+\beta} W_1,$$

  This is the optimal savings function in the first-period $W_2 = g(W_1)$, where in this case is a linear function. Note that in this case if $\beta = 0$, the consumer does not value consumption in the second period and the optimal savings for the next period is zero, that is $W_2 = 0$. We can compute the implied consumption decision rule for period 1 and 2

$$c_1^* = W_1 - W_2 = W_1 - \frac{\beta}{1+\beta} W_1$$

or

$$c_1^* = \frac{1}{1+\beta} W_1$$

and we can use the intertemporal constraint to compute second period consumption, that is $c_1 + c_2 = W_1$. But we already know the in the last period we have

$$c_2^* = W_2 = \frac{\beta}{1+\beta} W_1,$$

both consumption levels satisfy the resource constraint. Now we can compute the value function for the two-period problem.

$$v_2(W_1) = \ln(\frac{1}{1+\beta} W_1) + \beta \ln(\frac{\beta}{1+\beta} W_1)$$

rearranging terms we have

$$v_2(W_1) = A_2 + B_2 \ln(W_1)$$

where $A_2 = \ln(\frac{1}{1+\beta}) + \beta \ln(\frac{\beta}{1+\beta})$ and $B_2 = (1+\beta)$. It is important that the value function does not include the $max(\cdot)$ operator because we are using the optimal decisions to construct it.

- $T = 3$, now we solve

$$v_3(W_1) = \max_{W_2} \{\ln(W_1 - W_2) + \beta(A_2 + B_2 \ln(W_2))\}$$

the FOC imply

$$\frac{1}{(W_1 - W_2)} = \frac{\beta B_2}{W_2}$$

or

$$W_2 = \frac{\beta B_2}{1 + \beta B_2} W_1$$

but we know that $B_2 = 1 + \beta$, so we have,

$$W_2^* = \frac{\beta + \beta^2}{1 + \beta + \beta^2} W_1,$$

again, we have calculated the optimal decision rule given W_1. Now, we can compute the consumption levels for all three periods by using the sequential resource constraint

$$W_2 = W_1 - c_1$$
$$W_3 = W_2 - c_2$$
$$W_4 = W_3 - c_3$$

Then, we obtain

$$
\begin{aligned}
c_1^* &= \frac{1}{1+\beta+\beta^2}W_1 \\
c_2^* &= \frac{\beta}{1+\beta+\beta^2}W_1 \\
c_3^* &= \frac{\beta^2}{1+\beta+\beta^2}W_1
\end{aligned}
$$

Again, with the optimal decision rules we can compute the new value function

$$v_3(W_1) = A_3 + B_3 \ln(W_1)$$

where $A_3 = \ln(\frac{1}{1+\beta+\beta^2}) + \beta \ln(\frac{\beta}{1+\beta+\beta^2}) + \beta^2 \ln(\frac{\beta^2}{1+\beta+\beta^2})$ and $B_2 = (1+\beta+\beta^2)$.

However, there is an important aspect of the problem that we might have missed. The dynamic programming approach gives us recursive decision rules, that only depend on last period size of the cake. In particular, another way to look at decision rules is

$$
\begin{array}{lll}
W_2 = \frac{\beta+\beta^2}{1+\beta+\beta^2}W_1 & \Rightarrow & c_1 = \frac{1}{1+\beta+\beta^2}W_1 \\
W_3 = \frac{\beta}{1+\beta}W_2 & \Rightarrow & c_2 = \frac{1}{1+\beta}W_2 \\
W_4 = 0 \cdot W_3 & \Rightarrow & c_3 = W_3
\end{array}
$$

So when $\beta = 1$, we have

$$
\begin{array}{lll}
W_2 = \frac{2}{3}W_1 & \Rightarrow & c_1 = \frac{1}{3}W_1 \\
W_3 = \frac{1}{2}W_2 & \Rightarrow & c_2 = \frac{1}{2}W_2 \\
W_4 = 0 & \Rightarrow & c_3 = W_3
\end{array}
$$

If we want to have every thing in terms of first-period cake size, we only need to move across the optimal decision rules. In particular

$$
\begin{aligned}
c_1 &= \tfrac{1}{3}W_1 \\
c_2 &= \tfrac{1}{2}(\tfrac{2}{3}W_1) = \tfrac{1}{3}W_1 \\
c_3 &= (\tfrac{1}{2}(\tfrac{2}{3}W_1)) = \tfrac{1}{3}W_1
\end{aligned}
$$

As expected, with no discount it is optimal split the cake evenly over time. With a concave utility function this is optimal.