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Homework #2

1. A economy consists of two infintely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumers c^i at time t. Consumer i ranks consumptions streams by $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$ where $\beta \in (0, 1)$ and $u(c)$ is strictly increasing, concave, and twice continiously differentiable. Comsumer 1 is endowed with a stream of the consumption good $y_t^1 = 1, 0, 0, 1, 0, 0, 1, \ldots$. Consumer 2 is endowed with a stream of the consumption good $y_t^1 = 0, 1, 1, 0, 1, 1, 0, \ldots$. Assume that markets are complete with time-0 trading.

   (a) Define a competitive equilibrium.

   Definition An allocation for agent i is defined as state contingent function $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for $i = 1, 2$

   Definition A allocation is said to be a feasible allocation if it satisfies

$$\sum_{i=1}^{2} c_t^i(s^t) = \sum_{i=1}^{2} y_t^i(s_t) \tag{1}$$

   Definition A competitive equilibrium is a feasible allocation, $\{c^i\}_{i=1}^2 = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^2$, and a price system, $\{p_t^0(s^t)\}_{t=0}^{\infty}$, such that the allocation solves each household problem.

   (b) Compute a competitive equilibrum.
   For a given household i solves

$$U(c^i) = \max_{\{c_t^T(s^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

$$U(c^i) = \max_{\{c_t^T(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c_t^i(s^t)) \tag{2}$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) y_t^i(s_t), \quad \text{and} \quad c_t^1(s^t), c_t^2(s^t) \geq 0 \tag{3}$$

   First Order Conditions:

   $\beta^t \pi(s^t|s_0) u'(c_t^i(s^t)) = \gamma^i p_t^0(s^t)$

$$p_t^0(s^t) = \beta^t \pi(s^t|s_0) \frac{u'(c_t^i(s^t))}{\gamma^i} \tag{4}$$

   at $t = 0$, we have the following:

   $p_0^0(s^0) = \dfrac{u'(c_0^i(s^0))}{\gamma^i}$

   solving for $p_0^0(s^0) = 1$, we have the following:

   $1 = \dfrac{u'(c_0^i(s^0))}{\gamma^i} \quad \Rightarrow \quad \gamma^i = u'(c_0^i(s^0))$

   Subsititing into (4), obtain an system of price equations for the Arrow securities

$$p_t^0(s^t) = \beta^t \pi(s^t|s_0) \frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))} \tag{5}$$

   For the problem above for $i = 1, 2$, there is no aggregate risk or uncertainity (hence $c_t^i = c_0^i$), thus

   $\beta^t \pi(s^t|s_0) \dfrac{u'(c_t^1(s^t))}{u'(c_0^1(s^0))} = p_t^0(s^t) = \beta^t \pi(s^t|s_0) \dfrac{u'(c_t^2(s^t))}{u'(c_0^2(s^0))}$

In equilibrium, from $MRS^1_{0,T} = MRS^2_{0,T}$ obtains

$$\frac{u'(c^1_t(s^t))}{u'(c^1_0(s^0))} = \frac{u'(c^2_t(s^t))}{u'(c^2_0(s^0))} \tag{6}$$

Subsituting into the buget constraints (16) to find the feasible allocation for each agent for a competitive equilibrium.

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t\,\pi(s^t|s_0)\,\frac{u'(c^i_0(s^t))}{\gamma^i}\,[c^i_t(s^t) - y^i_t(s_t)] = 0$$

$$\frac{u'(c^i_0(s^t))}{\gamma^i}\sum_{t=0}^{\infty}\sum_{s^t} \beta^t\,\pi(s^t|s_0)\,[c^i_t(s^t) - y^i_t(s_t)] = 0$$

Because $\dfrac{u'(c^i_0(s^t))}{\gamma^i} \neq 0$, then obtain the following to solve for c^i_0

$$c^i_o \underbrace{\sum_{t=0}^{\infty}\beta^t}_{=\frac{1}{1-\beta}}\ \underbrace{\sum_{s^t}\pi(s^t|s_0)}_{=1} = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\,\pi(s^t|s_0)\,y^i_t(s_t)$$

Since the states are deterministics, the sum of probabilities in all states, (s^t has one element), $\pi(s^t|s_0) = 1\,\forall s^t$, and using a geometric series for for β

$$\frac{u'(c^i_0(s^t))}{\gamma^i}\sum_{t=0}^{\infty}\sum_{s^t} \beta^t\,\pi(s^t|s_0)\,[c^i_t(s^t) - y^i_t(s_t)] = 0$$

$$c^i_0 = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\,\pi(s^t|s_0)\,y^i_t(s_t) \tag{7}$$

Solve each consumption for each agent, $i = 1, 2$, where comsumer 1 is endowed with a stream of the consumption good $y^1_t = 1, 0, 0, 1, 0, 0, 1, \ldots$. Consumer 2 is endowed with a stream of the consumption good $y^1_t = 0, 1, 1, 0, 1, 1, 0, \ldots$.

agent 1	**agent 2**
$c^1 = (1-\beta)\sum_{t=0}^{\infty}\beta^t\,y^1_t$	$c^2 = (1-\beta)\sum_{t=0}^{\infty}\beta^t\,y^2_t$
$c^1 = (1-\beta)[\beta^0 + \beta^3 + \beta^6 + \ldots]$	$c^2 = (1-\beta)[\beta + \beta^2 + \beta^4 + \beta^5 + \ldots]$
$c^1 = (1-\beta)\sum_{t=0}^{\infty}\beta^{3t}$	$c^2 = (1-\beta)\sum_{t=0}^{\infty}[\beta^t - \beta^{3t+1}]$

using sum of a geometric series, can obtain:

$c^1 = \dfrac{(1-\beta)}{(1-\beta^3)} = \dfrac{1}{1+\beta+\beta^2}$	$c^2 = \dfrac{(1-\beta)}{(1-\beta)} - \dfrac{(1-\beta)}{(1-\beta^3)} = \dfrac{\beta\,(1+\beta)}{1+\beta+\beta^2}$

$$c^1 = \frac{1}{1+\beta+\beta^2} \qquad \text{and} \qquad c^2 = \frac{\beta\,(1+\beta)}{1+\beta+\beta^2} \tag{8}$$

$$\text{subject to:}\quad \sum_{1}^{2} c^i = w_0 = 1 \Rightarrow c^1 + c^2 = 1 \tag{9}$$

(c) Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of the asset be?

From Cochrane, (pg.52 1ed.), using to what he refers to as the **happy-meal theorem (3.1):**

$$p(x) = \sum_{s} pc(s)x(s) \tag{10}$$

where $x(s)$ denotes an asset's payoff in state of nature s, pc is the price of a contingent claim and (s) is used to denote which state s the claim pays off. From (10), obtain

$$p = \sum_{s} \beta^t \frac{u'(c_{t+1}))}{u'(c_t)} \pi(s^t|s_0)\, x_{t+1} \tag{11}$$

In the setup of problem #1: $\dfrac{u'(c_{t+1}))}{u'(c_t)} = 1$, and $\pi(s^t|s_0) = 1$. Obtain:

$$p = \sum_{s} \beta^t\, x_{t+1} \tag{12}$$

Since a derivative asset that promises to pay .05 units of consumption each period,

$$p_t^0(s) = \sum_{s} \beta^t\, (.05) = \frac{.05}{1-\beta}$$

2. Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowement obeying, $y_{t+1} = \lambda_{t+1}\, y_t$, where y_t is the endowment at time t and $\{\lambda_{t+1}\}$ is governed by a two-state Markov chain with transition matrix, $P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}$, and intial distribution $\pi_\lambda = [\pi_0, 1-\pi_0]$. The value of λ_t is given by $\overline{\lambda_1} = 0.98$ in state 1 and $\overline{\lambda_2} = 1.03$ in state 2. Assume that the history of y_s and λ_s, up to t is observed at time t. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by $E_0 \sum_{t=0}^{\infty} \beta^t \, \dfrac{c^{1-\gamma}}{1-\gamma}$, here $\beta \in (0,1)$ and $\gamma \geq 1$.

(a) Define a competitive equilibrium.

Definition A allocation is said to be a <u>feasible allocation</u> if it satisfies

$$\sum_{i=1}^{2} c_t^i(s^t) = \sum_{i=1}^{2} y_t^i(s_t) \tag{13}$$

$$\text{Since one agent} \rightarrow c_t(s^t) = y_t(s_t) \tag{14}$$

Definition A <u>competitive equilibrium</u> is a feasible allocation, $\{c^i\}_{i=1}^2 = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^2$, and a price system, $\{p_t^0(s^t)\}_{t=0}^{\infty}$, such that the allocation solves each household problem.

For a given household i solves

$$U(c^i) = \max_{\{c_t^T(s^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t\, u(c_t^i)$$

$$U(c^i) = \max_{\{c_t^T(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t\, \pi(s^t|s_0)\, u(c_t^i(s^t)) \tag{15}$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) y_t^i(s_t), \quad \text{and} \quad c_t^1(s^t), c_t^2(s^t) \geq 0 \tag{16}$$

(b) From (14), **consumption** = **endowment**, since only one agent obtain the following:

$$c_t = y_t \tag{17}$$

In (17), y is defined as $y_{t+1} = \lambda_{t+1}\, y_t$, where λ_{t+1} is defined as a two state Markov Chain. Using the properites of Markov process,

$$y_{t+1} = \prod_{i=0}^{t} \lambda_i\, y_0 \tag{18}$$

Substituting (17) into the first order conditions of the household problem, into (5), will obtain an equation that will solve the price system for any asset that will lead to competitive equilibrium.

$$p_t^0(s^t) = \beta^t\, \pi(s^t|s_0)\, \frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))} \qquad \text{from (5)} \tag{19}$$

$$p_t^0(s^t) = \beta^t\, \pi(s^t|s_0)\, \frac{u'(y_t)}{u'(y_0)} \qquad \text{using (17)} \tag{20}$$

Suppose $p_{11} = .8, p_{22} = .85, \pi_0 = .5, \beta = .96$, and $\gamma = 2$. Suppose economy starts off with $\lambda_0 = .98$ and $y_0 = 1$.

Also, $P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .15 & .85 \end{bmatrix}$ and intial distribution $\pi_\lambda = [\pi_0, 1-\pi_0] = [0.5, .5]$. The value of λ_t is given by $\overline{\lambda_1} = 0.98$ in state 1 and $\overline{\lambda_2} = 1.03$ in state 2.

(c) I will use the given conditions in the setup of the problem, stated above, and using the properties of a Markov chain, (that is, as time $\to \infty$, the probability of each state occuring will converge to a stationary distrubition) to solve for the average growth rate of consumption.

Goal: Find a stationary distribution such that,

$$\sum a_i = 1 \text{ and } a = aP, \text{ where } a = [\alpha, 1-\alpha] \tag{21}$$

Using P, setup a system of equations to solve for α.

$$a = aP \longrightarrow [\alpha, 1-\alpha] = [\alpha, 1-\alpha] \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}$$

From top equation, (Eq1) and **From bottom equation, (Eq2)**

$$\begin{aligned} \alpha &= \alpha\, p_{11} + (1-\alpha)(1-p_{22}) \\ \alpha &= \alpha\, p_{11} + 1 - \alpha - p_{22} + \alpha\, p_{22} \\ \alpha + \alpha - \alpha\, p_{11} - \alpha\, p_{22} &= 1 - p_{22} \\ \alpha(2 - p_{11} - p_{22}) &= 1 - p_{22} \\ \alpha &= \frac{1-p_{22}}{(2-p_{11}-p_{22})} \end{aligned}$$

$$\begin{aligned} (1-\alpha) &= \alpha\,(1-p_{11}) + (1-\alpha)p_{22} \\ (1-\alpha) + (1-\alpha)p_{22} &= \alpha\,(1-p_{11}) \\ (1-\alpha)(1-p_{22}) &= \alpha\,(1-p_{11}) \\ \frac{(1-\alpha)}{\alpha} &= \frac{(1-p_{11})}{(1-p_{22})} \\ \frac{1}{\alpha} &= \frac{(1-p_{11})}{(1-p_{22})} + 1 = \frac{(1-p_{11}+1-p_{22})}{(1-p_{22})} \end{aligned}$$

Solving in both equations for α obtain equalvalent realtions, which is suppose to happen since it is a stationary distribution.

$$\textbf{From Eq1: } \alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} \qquad \textbf{From Eq2: } \alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} \tag{22}$$

Therefore subsituting the values for $p_{11} = .8$ and $p_{22} = .85$ in (22) will obtain:

$$\alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} = \frac{1-.85}{2-.80-.85} = \frac{3}{7}$$

$$a = [\alpha, 1-\alpha] = \left[\frac{3}{7}, \frac{4}{7}\right] \tag{23}$$

By properties of Markov process, as time $\to \infty$, the probability of the each state occuring will converge to the corresponding probabilities in the stationary distribution. Hence:

$$P(\lambda_t = \overline{\lambda_i}\,) = a \text{ where i=1,2} \tag{24}$$

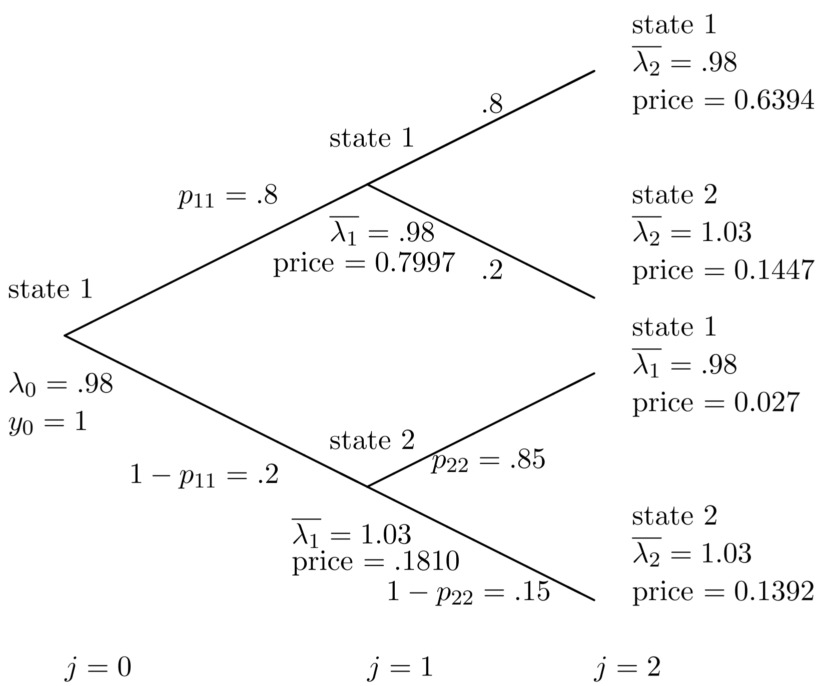
Therefore will obtain

$$P(\lambda_t = \overline{\lambda_1}\,) = \frac{3}{7} \quad \text{and} \quad P(\lambda_t = \overline{\lambda_2}\,) = \frac{4}{7} \tag{25}$$

Using (25), able to compute the average growth rate of consumption, before having observed λ_0.

$$\begin{aligned} E[\lambda] &= P(\lambda_t = \overline{\lambda_1}\,)\overline{\lambda_1} + P(\lambda_t = \overline{\lambda_2}\,)\overline{\lambda_2} \\ &= \left(\frac{3}{7}\right)(.98) + \left(\frac{4}{7}\right)(1.03) \\ &= 1.0086 \end{aligned} \tag{26}$$

In order to compute the time-0 prices for the for each of the three bonds promising to pay one unit of time-$j = 0, 1, 2$.
Assume that given intial state, $\lambda_0 = .98 = \overline{\lambda_1}$. Thus, has corresponding values, $p_{11} = .8$, and $1 - p_{11} = .2$ on first branch of tree.



From (20), using the following given information, $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, where $\gamma = 2$, $\beta = .96$, and $y_0 = 1$.
Also known from (18), $y_t = \prod_{i=1}^{t} \lambda_i \, y_0 \quad \rightarrow \quad y_t = \prod_{i=1}^{t} \lambda_i$

From the given conditions, will obtain

$$\begin{aligned} p_t^0(s^t) &= \beta^t \, \pi(s^t|s_0) \, \frac{u'(y_t)}{u'(y_0)} \\ p_t^0(s^t) &= (.96)^t \, \pi(s^t|s_0) \left(\frac{y_0}{y_t}\right)^2 \end{aligned} \tag{27}$$

I will use (27), to compute the price at each time $t = 0, 1, 2$

t=1

- $p^0_1(\overline{s^1}|s_0) = (.96)\,(.8)\,\left(\frac{1}{.98}\right)^2 = .7997$
- $p^0_1(\underline{s^1}|s_0) = (.96)\,(.2)\,\left(\frac{1}{1.03}\right)^2 = .1810$

t=2

- $p^0_2(\overline{s^2}|s_0) = (.96)^2\,(.8)(.8)\,\left(\frac{1}{(.98)(.98)}\right)^2 = .6394$
- $p^0_2(\underline{\overline{s^2}}|s_0) = (.96)^2\,(.2)(.15)\,\left(\frac{1}{(1.03)(.98)}\right)^2 = .027$
- $p^0_2(\underline{s^2}|s_0) = (.96)^2\,(.8)(.2)\,\left(\frac{1}{(1.03)(.98)}\right)^2 = .1447$
- $p^0_2(\underline{\underline{s^2}}|s_0) = (.96)^2\,(.85)(.15)\,\left(\frac{1}{(1.03)(1.03)}\right)^2 = .1392$

From Cochrane, (pg.52 1ed.), using to what he refers to as the **happy-meal theorem (3.1):**

$$p(x) = \sum_s pc(s)x(s) \tag{28}$$

where $x(s)$ denotes an asset's payoff in state of nature s, pc is the price of a contingent claim and (s) is used to denote which state s the claim pays off. In problems **(d)-(f)**, the bond promises to pay one unit, $\mathbf{x(s) = 1}$. From (28), able to compute the price of the bond promising to pay one unit of time-j comsumption for $j = 0, 1, 2$.

(d) Compute the time-0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time-j consumption for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 1$

t= 1

- $p(x) = .7997 + .1810 = .9807$

t= 2

- $p(x) = .6394 + .027 + .1447 + .1392 = .9508$

(e) Compute the time-0 prices of three bonds, in particular, those promising to pay one unit of time-j contingent consumption on $\lambda_j = \overline{\lambda_1}$ for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 1 \quad$ from $\lambda_0 = \overline{\lambda_1}$

t= 1

- $p(x) = .7997$

t= 2

- $p(x) = .6394 + .027 = .666$

(f) Compute the time-0 prices of three bonds, in particular, those promising to pay one unit of time-j contingent consumption on $\lambda_j = \overline{\lambda_2}$ for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 0$ from $\lambda_0 = \overline{\lambda_1}$

t= 1

- $p(x) = .1810$

t= 2

- $p(x) = .1447 + .1392 = .2839$

(g) Compare the prices that you computed in parts **(d)-(f)**.

From above, can see that at each time t, the sum of each price in answers **(e)** and **(f)**, is equal to the price of the risk free discount bond, **(d)**.

At each time, $t = 0, 1, 2$ **(e)+(f)=(d)**.

3. An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer. The endowments to consumers $i = 1, 2$, are $y_t^1 = s_t$ and $y_t^2 = 1$, where s_t is a random variable governed by a two-state Markov chain with values $s_t = \overline{s_1} = 0$ or $s_t = \overline{s_2} = 1$. The Markov chain has time-invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the intial state is $\pi_0(s)$. The aggregate endowment at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) \, \ln[c_t^i(s^t)] \tag{29}$$

where $\pi(s^t)$ is the probability of the history

$$s^t = (s_0, s_1, \ldots, s_t)$$

(a) Give a formula for $\pi(s^t)$

From properities of conditional expectation, can obtain

$$\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_0) \tag{30}$$

(b) Let $\theta \in (0, 1)$ be a Pareto weight on household 1. Consider the **social planner problem**

$$\max_{c_t^1(s^t), c_t^2(s^t)} \left\{ \theta \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) \, \ln[c_t^1(s^t)] + (1-\theta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) \, \ln[c_t^2(s^t)] \right\} \tag{31}$$

$$\begin{aligned} \text{subject to} \quad c_t^1(s^t) + c_t^2(s^t) &= Y(s^t) \\ c_t^i(s^t) &\geq 0 \end{aligned} \tag{32}$$

Solve the Pareto problem, taking θ as a parameter

To be consistent with the lecture notes, make the following subsitution, $\lambda^1 = \theta$ and $\lambda^2 = (1 - \theta)$. Thus, $\lambda^i \in (0, 1)$ and $\sum_{i=1}^{2} \lambda^i = 1$

Using the above subsitution, (31) can be rewitten as:

$$\max_{\{c_t^1(s^t)\,,\, c_t^2(s^t)\}} \left\{ \underbrace{\lambda^1 \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) \, \ln[c_t^1(s^t)]}_{\text{for agent 1}} + \underbrace{\lambda^2 \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) \, \ln[c_t^2(s^t)]}_{\text{for agent 2}} \right\} \tag{33}$$

$$\begin{aligned} \text{subject to} \quad c_t^1(s^t) + c_t^2(s^t) &= Y(s^t) \\ c_t^i(s^t) &\geq 0 \end{aligned} \tag{34}$$

Solving using Legrange Multipliers, will obtain

$$L = \sum_{i=1}^{2}\sum_{t=0}^{\infty}\sum_{s^t} \lambda^i \,\beta^t \,\pi(s^t|s_0)\, u(c^i(s^t)) - \sum_{t=0}^{\infty} \gamma_t \left(\sum_{i=1}^{2}(c_t^i(s^t) - Y(s^t)\right) \tag{35}$$

<u>First Order Conditions (35)</u>

$$\left[c_t^i(s^t)\right]: \qquad \lambda^i \,\beta^t \,\pi(s^t|s_0)\, u'(c^i(s^t)) - \gamma_t = 0$$

For two different consumers, i and j, obtain

$$\frac{u'(c^{\,i}(s^t))}{u'(c^{\,j}(s^t))} = \frac{\lambda^j}{\lambda^i} \tag{36}$$

From setup of problem, $u(c_t^i(s^t)) = \ln(c_t^i(s^t)) \longrightarrow \quad u'(c_t^i(s^t)) = \dfrac{1}{c_t^i(s^t)}$, for each agent $i = 1, 2$ Subsitute this into (36), obtain

$$\begin{aligned} \frac{c^{\,j}(s^t)}{c^{\,i}(s^t)} &= \frac{\lambda^j}{\lambda^i} \\ \frac{c^{\,2}(s^t)}{c^{\,1}(s^t)} = \frac{\lambda^2}{\lambda^1} &= \frac{1-\theta}{\theta} \\ \frac{c^{\,2}(s^t)}{c^{\,1}(s^t)} &= \frac{1-\theta}{\theta} \end{aligned} \tag{37}$$

Similiar to Homework 1, subsitute (37) into the budget constraints (34).

Obtains the optimal sequence of $c^{\,i}(s^t)$ to each agent which will solve the optimal Pareto problem.

$$c_t^1(s^t) + c_t^2(s^t) = \left(1 + \frac{1-\theta}{\theta}\right) c_t^1(s^t) = Y(s^t) \tag{38}$$

In a symmetric allocation, $\lambda^i = \lambda^j$ all agents recieve the same allocation. Solve for each agents consumption allocation, will obtain

$$\begin{aligned} c_{\,t}^{\,1}(s^t) &= \lambda^1\, Y(s^t) = \theta Y(s^t) \\ c_{\,t}^{\,2}(s^t) &= \lambda^2\, Y(s^t) = (1-\theta) Y(s^t) \end{aligned} \tag{39}$$

(c) Define the competitive equilibrium with history-dependent Arrow-Debreu securities traded once and for all at time 0. Becareful to define all of the objects that compose a competitive equilibrium.

Definition A allocation $\left\{\{c_t^i s^t\}_{t=0}^{\infty}\right\}_{i=1}^{\infty}$is said to be a <u>feasible allocation</u> if it satisfies

$$\sum_{i=1}^{I} c_t^i(s^t) = Y(s_t), \quad \forall t \tag{40}$$

Definition A <u>competitive equilibrium</u> is a feasible allocation, $\{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^{2}$,

and a price system, $\{p_t^0(s^t)\}_{t=0}^{\infty}$, such that the allocation solves each household problem.

(d) Compute the competitive equilibrium price system.
(i.e., Find the prices of all of the Arrow-Debreu securities)
As in the previous problems #1 − 2, if we would like to determine the price of the asset at any time t, need a pricing equation, from (5).

$$\begin{aligned} p_t^0(s^t) &= \beta^t\,\pi(s^t|s_0)\,\frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))} && \text{from (5)} \\ &= \beta^t\,\pi(s^t|s_0)\,\frac{u'(\lambda^i\,Y(s^t))}{u'(\lambda^i\,Y(s^0))} && \text{subsitute in (39)} \\ p_t^0(s^t) &= \beta^t\,\pi(s^t|s_0)\,\frac{Y(s^0)}{Y(s^t)} && (41) \end{aligned}$$

Let $Y(s^t) = g(s^t)Y(s_0)$, from setup of problem. Then, (41), can be rewitten using this subsitution.

$$p_t^0(s^t) = \frac{\beta^t\,\pi(s^t|s_0)}{g(s^t)} \tag{42}$$

(e) Tell the relationship between the solutions (indexed by θ) of the Parteo problem and the competitive equilibrium allocation.

By the First and Second Welfare theorem, the prices obtained in the competitive equilibrium market are the **same** as the prices obtained in the soical planner's problem.

From the **First Welfare Theorem**, a (competitive) equilibrium allocation is Pareto efficient.

In the Pareto problem, the social planner is to find a set of parteo efficient allocation that maximizes each household utility. Even though each agent may or may not have a different weight, (designated by the social planner), this price is independent of each agent's weight.

(f) Briefly tell how you can compute the competitive equilibrium price system before you have figured out the competitive equilibrium allocation.

The competitive equilibrium price system can be solved using (42). Note that the price system is independent of each individual's consumptions. The price is computed from the growth rate of the aggregate endowment. Thus, determine the growth rate and we can solve the price system, even though a feasible allocation has been not defined.

(g) Define a recursive competitive equilibrium with trading every period in one-period Arrow securities only. Describe all of the objects of which such an equilibrium is composed. (Please denominate the prices of one-period time-$t+1$ state contingent Arrow securities in units of time-t consumption.)

A **recursive competitive equilibrium** is an intial distribution of assets, $a_0(s_0)$, given the feasible allocations $\{c^i(a,s), a^i_(a, s^t, s_{t+1})\}$ and price system $\sum_{s^{t+1}} Q_t(s_{t+1}|s^t)$

The allocation solves each household problem,

$$\max \sum_{t=0}^{\infty} \beta^t\, u(c_t^i(s^t))\,\pi(s^t) \tag{43}$$

$$\text{subject to } \; c_t^i(s^t) + \sum_{s^t} Q_t(s^t) a_{t+1}^i(s^{t+1}|s_t) \le y_t^i(s^t) + a^i_(s^t)$$

$$-a_{t+1}^i(s^{t+1}|s_t) \le A_{t+1}^i(s^t)$$

$$\text{where } \; A_{t+1}^i(s^t) = \sum_{\tau\ge t}\sum_{s^\tau|s^t} Q_t(s^\tau|s_t)\, y_t^i(s^\tau|s_t)$$

(h) (i) Tell how to compute the prices of one-period Arrow securities. How many prices are there? (i.e., how many numbers do you have to compute?)

(ii) Compute all of these prices in the special case that $\beta = .95$, and $\pi(s_j|s_i) = P_{i,j}$, where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$

Answer(i) Note: The sequential market structure attains the same consumption allocation as the the competitive equilibrium with Arrow-Debreu structure.

In this economy, a sequential equilibrium is an allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s_t), b_{t+1}^i(s_{t+1})\}_{t=0}^\infty\}_{i=1}^I$ and a price system $\{Q(s_{t+1}|s_t)\}_{t=0}^\infty$ such that the allocation solves each household problem, and

$$U(c^i) = \max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) u(c_t^i(s^t)) \tag{44}$$

$$\text{subject to} \quad c^i(s^t) + \sum_{s^{t+1}} Q(s_{t+1}|s_t) b_{t+1}^i(s_{t+1}) = y_t^i(s_t) + b_t^i(s_t) \;\; \forall s \tag{45}$$

Solving using Legrange Multipliers, will obtain

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \pi(s^t|s_0) u(c_t^i(s^t)) - \sum_{t=0}^{\infty} \gamma_t \left(c_t^i(s^t) + \sum_{s^{t+1}} Q(s_{t+1}|s_t) b_{t+1}^i(s_{t+1}) - \left(y_t^i(s_t) + b_t^i(s_t)\right) \right) \tag{46}$$

<u>First Order Conditions (44)</u>

$$\left[c_t^i(s^t)\right] : \quad \beta^t \, \pi(s^t|s_0) u'(c_t^i(s_t)) - \gamma_t = 0 \quad \Longrightarrow \quad \gamma_t = \beta^t \, \pi(s^t|s_0) u'(c_t^i(s^t))$$

$$\left[b_{t+1}^i\right] : \quad \gamma_t \, Q(s_{t+1}|s_t) - \gamma_{t+1} = 0 \quad \Longrightarrow \quad Q(s_{t+1}|s_t) = \frac{\gamma_{t+1}}{\gamma_t}$$

Therefore, using subsituting the above equations, will obtain how to compute one-period Arrow securities:

$$\begin{aligned} Q(s_{t+1}|s_t) &= \frac{\gamma_{t+1}}{\gamma_t} = \frac{\beta^{t+1} \, \pi(s^{t+1}|s_0) u'(c_t^i(s_{t+1}))}{\beta^t \, \pi(s^t|s_0) u'(c_t^i(s_t)} \\ Q(s_{t+1}|s_t) &= \beta \, \pi(s^{t+1}|s_t) \frac{u'(c_t^i(s_{t+1}))}{u'(c_t^i(s_t)} && (47) \\ &= \beta \, \pi(s^{t+1}|s_t) \frac{(1 + s_t)}{(1 + s_{t+1})} && (48) \end{aligned}$$

How many prices are there? There are 4^n one-period arrow securities.

Answer (ii) Compute all of these prices in the special case that $\beta = .95$, and $\pi(s_j|s_i) = P_{i,j}$, where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$

In order to compute the compute the equilibrium prices of the security, using the given conditions above and subsituting in (48), will obtain:

$$\begin{aligned} Q(s_1|s_1) &= (.95)(.8)\frac{1+0}{1+0} = .76 \\ Q(s_2|s_1) &= (.95)(.2)\frac{1+0}{1+1} = .095 \\ Q(s_1|s_2) &= (.95)(.3)\frac{1+1}{1+0} = .57 \\ Q(s_2|s_2) &= (.95)(.7)\frac{1+1}{1+1} = .665 \end{aligned}$$

(i) Within the one-period Arrow economy, a new asset is produced. One of the households decides to market a one-period-ahead riskless claim to one unit of consumption (a one-period real bill). Compute the equilibrium prices of this security when $s^t = 0$ and when $s^t = 1$. Justify your formula for these prices in terms of first principles.

$$[\text{at } s_t = 0] \qquad P = .76 + .095 = .855 \tag{49}$$

$$[\text{at } s_t = 1] \qquad P = .57 + .665 = 1.235 \tag{50}$$

(j) Within the one-period Arrow securities equilibrium, a new is introduced. One of the households decides to market a two-period-ahead riskless claim to one unit of consumption (a two-period real bill). Compute the equilibrium prices of this security when $s^t = 0$ and when $s^t = 1$.

In order to compute prices of this security in a market with a two-period-ahead riskless claim, need a formula to price an asset j-step ahead.

$$Q_j(s_{t+2}|s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)Q_{j-1}(s_{t+j}|s_{t+1}) \tag{51}$$

From (51), in **j**, let $j = 2$, thus obtain:

$$Q_2(s_{t+2}|s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)Q_1(s_{t+2}|s_{t+1}) \tag{52}$$

From (52), need to calculate $Q_2 = Q_1^2$

$$Q_2 = Q_1^2 = \begin{bmatrix} .76 & .095 \\ .57 & .665 \end{bmatrix}^2 = \begin{bmatrix} .6318 & .1354 \\ .8123 & .4964 \end{bmatrix}$$

$$[\text{at } s_t = 0] \qquad P = Q^2(1,1) + Q^2(1,2) = .7671 \tag{53}$$

$$[\text{at } s_t = 1] \qquad P = Q^2(2,1) + Q^2(2,2) = 1.309 \tag{54}$$

(k) Within the one-period Arrow securities equilibrium, a new is introduced. One of the households decides at time t to market a five-period-ahead claims to consumption at $t+5$ contingent on the value of s_{t+5}. Compute the equilibrium prices of this security when $s^t = 0$ and $s^t = 1$, $s^{t+5} = 0$ and $s^{t+5} = 1$. From (51), in **j**, let $j = 5$, thus obtain:

$$Q_5(s_{t+5}|s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)Q_1(s_{t+5}|s_{t+1}) \tag{55}$$

From (55), need to calculate $Q_5 = Q_1^5$

$$Q_5 = Q_1^5 = \begin{bmatrix} .76 & .095 \\ .57 & .665 \end{bmatrix}^5 = \begin{bmatrix} .4740 & .15 \\ .8996 & .3740 \end{bmatrix}$$

$$\begin{aligned}
Q_5(s_{t+5} = 0|s_t = 0) &= .4740 \\
Q_5(s_{t+5} = 1|s_t = 0) &= .15 \\
Q_5(s_{t+5} = 0|s_t = 1) &= .8996 \\
Q_5(s_{t+5} = 1|s_t = 0) &= .3240
\end{aligned}$$