Jaime Frade
ECO5282-Dr. Garriga
Homework #3

1. Consider and economy with a representative consumer with preferences described by $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $u(c_t) = \ln(c_t + \gamma)$ where $\gamma \geq 0$ and c_t denotes consumption of the fruit in period t. The sole source of the single good is an everlasting tree that produces d_t units of the consumption good in period t. The dividend process d_t is Markov, with $prob\{d_{t+1} \leq d' | d_t = d\} = F(d', d)$. Assume the conditional density $f(d', d)$ of F exists. There are competitive markets in the title of trees and in state-contingent claims. Let p_t be the price at t of a title to all future dividends from the tree.

(a) Prove that the equilibrium prices p_t satisfies

$$p_t = (d_t + \gamma) \sum_{j=1}^{\infty} \beta^t E_t \left(\frac{d_{i+j}}{d_{i+j} + \gamma} \right) \tag{1}$$

For a given household, the comsumer solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

Subject to the following budget constraint

$$A_{t+1} = R_t(A_t + y_t - c_t) \tag{3}$$

From (3),

$$\begin{aligned}
c_t &\implies \text{consumption of an agent at time } t \\
y_t &\implies \text{agent's labor income} \\
A_t &\implies \text{amount of a single asset valued in the units of the} \\
& \text{consumption good held at start of time } t \\
& \text{(Note: } A_0 \text{ is given)} \\
R_t &\implies \text{real gross rate of return on asset between time } t \text{ and } t+1
\end{aligned}$$

By using the Legrange multipliers, combining the first order conditions in (2) we obtain the **Euler equation** that is necessary condition of optimality for $t = 0, 1, \ldots$.

$$u'(c_t) = E_t \beta R_t u'(c_{t+1}) \tag{4}$$

However (4) does not applicable to models with complete general equilibrium. In the Lucas's asset-pricing model, general equilibrium is used. This model assumes that the labor income is zero, $y_t = 0$, $\forall\ t$. It also assumes that the only durable good in the economy is a set of trees, (which equal the number of people in economy). The tree also yields fruit (dividends) in the amount of d_t to its owner ar the beginning of time t. This fruit is also nonstorable. Each consumer starts a $t = 0$ with, one tree and initial dividend of good, (fruit).

$$\begin{aligned}
\text{Let } p_t &\implies \text{price of a tree at time } t \\
R_t &= \frac{p_{t+1} + d_{t+1}}{p_t}
\end{aligned} \tag{5}$$

Using (5) subsitute into (4), will obtain the following:

$$1 = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(\frac{p_{t+1} + d_{t+1}}{p_t} \right) \tag{6}$$

In this economy, equilibrium must exist because all consumers are identical, the utility function is strictly increasing, and there is only one place for goods, period's dividends. Therefore obtain:

$$c_t = d_t \tag{7}$$

Substituting (7) into (6) and solve for p_t, will obtain the following:

$$p_t \;=\; E_t\,\beta\,\frac{u'(d_{t+1})}{u'(d_t)}\,(p_{t+1} + d_{t+1}) \tag{8}$$

Given $u(c_t) = \ln(c_t + \gamma)$ where $\gamma \geq 0$, and using (7) and differentiating to plug into (8) will obtain:

$$p_t = E_t\,\beta\,\frac{(d_t + \gamma)}{(d_{t+1} + \gamma)}\left(\underbrace{p_{t+1}}_{\Uparrow} + d_{t+1}\right) \tag{9}$$

Using recursions on (9) for the value p_{t+1}, will obtain:

$$p_t \;=\; E_t\left[\beta\,\frac{(d_t + \gamma)}{(d_{t+1} + \gamma)}\left(d_{t+1} + \underbrace{\beta\,\frac{(d_{t+1} + \gamma)}{(d_{t+2} + \gamma)}\,(p_{t+2} + d_{t+2})}_{=p_{t+1}}\right)\right]$$

$$p_t \;=\; E_t\left[\beta\,\frac{(d_t + \gamma)}{(d_{t+1} + \gamma)}\left(d_{t+1} + \beta\,\frac{(d_{t+1} + \gamma)}{(d_{t+2} + \gamma)}\left(d_{t+2} + \beta\,\frac{(d_{t+2} + \gamma)}{(d_{t+3} + \gamma)}\,(d_{t+3} + p_{t+3})\right)\right)\right]$$

$$p_t \;=\; E_t\left[\beta\,\frac{(d_t + \gamma)}{(d_{t+1} + \gamma)}d_{t+1} + \beta^2\,\frac{(d_t + \gamma)}{(d_{t+2} + \gamma)}d_{t+2} + \beta^3\,\frac{(d_t + \gamma)}{(d_{t+3} + \gamma)}d_{t+3} + \cdots\right]$$

From above equation, can see a pattern forming, obtain

$$p_t \;=\; (d_t + \gamma)E_t\sum_{j=0}^{\infty}\left[\beta^j\left(\frac{d_{t+j}}{d_{t+j} + \gamma}\right)\right] + \lim_{j\to\infty} E_t\left[\beta^j\left(\frac{d_t + \gamma}{d_{t+j} + \gamma}\right)p_{t+j}\right]$$

As $j \to \infty$ $\lim \to 0$, will prove result for pricing equation.

$$p_t \;=\; (d_t + \gamma)E_t\sum_{j=0}^{\infty}\beta^j\left(\frac{d_{t+j}}{d_{t+j} + \gamma}\right) \tag{10}$$

(b) Modify the Lucas Model to a market that consist of one- and two period perfectly safe loans, which bear gross rates of return R_{1t} and R_{2t}. R_{1t}^{-1} is the price of a perfectly sure claim to one unit of consumption at time $(t+1)$. R_{2t}^{-1} is the price of a perfectly sure claim to one unit of consumption at time $(t+2)$.
The agent needs to solve the following maximum

$$\max_{c_t, L_{1t+1}, L_{2t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{11}$$

Subject to the following budget constraint, where L_{jt} is the amount lent for j period at time t.

$$c_t + L_{1t} + L_{2t} \leq d_t + R_{1t-1}L_{1t-1} + R_{2t-2}L_{2t-2} \tag{12}$$

Solving (11), using Legrange Multiplier, where $\{\lambda_t\}$ is a sequence of random Legrange Multipliers. Will obtain the following:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t(d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2} - c_t - L_{1t} - L_{2t})\right] \tag{13}$$

First Order Conditions (11) with respect to $\{c_t, L_{1t}, L_{2t}\}$. Also $E_0 = E_t$ since the Markov property.

$$\begin{array}{llcl}
[c_t]: & E_0\beta^t(u'(c_t) - \lambda_t) = 0 & \Longrightarrow & \lambda_t = u'(c_t) \\
[L_{1t}]: & E_0\beta^t(\beta\lambda_{t+1}R_{1t} - \lambda_t) = 0 & \Longrightarrow & \lambda_t = E_t(\beta\lambda_{t+1}R_{1t}) \\
[L_{2t}]: & E_0\beta^t(\beta^2\lambda_{t+2}R_{2t} - \lambda_t) = 0 & \Longrightarrow & \lambda_t = E_t(\beta^2\lambda_{t+2}R_{2t})
\end{array}$$

Combining the first order conditions above will obtain the following **Euler Equations**:

$$E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}R_{1t}\right] = 1 \tag{14}$$

$$E_t\left[\beta^2\frac{u'(c_{t+2})}{u'(c_t)}R_{2t}\right] = 1 \tag{15}$$

Given that both assets are riskless, rewrite (14) and (15) as:

$$R_{1t}^{-1} = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\right] \tag{16}$$

$$R_{2t}^{-1} = E_t\left[\beta^2\frac{u'(c_{t+2})}{u'(c_t)}\right] \tag{17}$$

(i) Find a formula for the risk-free one-period interest rate R_{1t}.

From (16), using given $u(c_t) = \ln(c_t + \gamma)$ and (7) in equilibrium, will obtain a formula for th risk-free one-period interest rate R_{1t}.

$$R_{1t}^{-1} = E_t\left[\beta\frac{(d_t+\gamma)}{d_{t+1}+\gamma}\right] \tag{18}$$

(ii) Prove that in the special case in which $\{d_t\}$ is independently and identically distributed, R_{1t} is given by $R_{1t}^{-1} = \beta\, k\,(d_t + \gamma)$, where k is a constant.

If assume that $\{d_t\}$ is $i.i.d$, then (18) can be rewritten:

$$\begin{aligned}
R_{1t}^{-1} &= (d_t+\gamma)\beta\, E_t\left[\frac{1}{d_{t+1}+\gamma}\right] \\
R_{1t}^{-1} &= (d_t+\gamma)\beta\, E_t\left[\frac{1}{d_{t+1}+\gamma}\right] \quad \Longrightarrow \text{Let } \; k = E_t\frac{1}{d_{t+1}+\gamma} \\
R_{1t}^{-1} &= (d_t+\gamma)\beta\, k
\end{aligned} \tag{19}$$

(iii) Give a formula for k.

From (19), and assumption that $\{d_t\}$ is $i.i.d$, obtain a formula for k, which is not a function of time:

$$k = E_t\left[\frac{1}{d_{t+1}+\gamma}\right] \tag{20}$$

(c) (i) Find a formula for the risk-free two-period interest rate R_{2t}.

From (17), using given $u(c_t) = \ln(c_t + \gamma)$ and (7) in equilibrium, will obtain a formula for th risk-free one-period interest rate R_{2t}.

$$R_{2t}^{-1} \quad = \quad E_t\left[\beta^2 \frac{(d_t+\gamma)}{d_{t+2}+\gamma}\right] \tag{21}$$

(ii) Prove that in the special case in which $\{d_t\}$ is independently and identically distributed, R_{2t} is given by $R_{2t}^{-1} = \beta^2\, k\,(d_t+\gamma)$, where k is the same constant you found in part **(b)**.

If assume that $\{d_t\}$ is $i.i.d$, then (18) can be rewritten:

$$\begin{aligned}
R_{2t}^{-1} &= (d_t+\gamma)\beta^2\, E_t\left[\frac{1}{d_{t+2}+\gamma}\right] \\
R_{2t}^{-1} &= (d_t+\gamma)\beta^2\, E_t\left[\frac{1}{d_{t+2}+\gamma}\right] \qquad \Longrightarrow \text{Let } \; j = E_t\frac{1}{d_{t+2}+\gamma} \\
R_{2t}^{-1} &= (d_t+\gamma)\beta^2\, j
\end{aligned} \tag{22}$$

- Prove that j from (22) is same k from (20). From $\{d_t\}$ is $i.i.d$ and given that the dividend process d_t is Markov, with $prob\{d_{t+1} \leq d' | d_t = d\} = F(d', d)$.

$$\begin{aligned}
j &= E_t\left[\frac{1}{d_{t+2}+\gamma}\right] \\
&= E_{t+1}\left[\frac{1}{d_{t+2}+\gamma}\right] \\
&= k
\end{aligned}$$

2. Consider the following version of the Lucas's tree economy. There are two kinds of trees. The first kind is ugly and gives no direct utility to consumers, but yields a stream of fruit $\{d_{1t}\}$, where d_{1t} denotes a positive random process obeying a first-order Markov process. The second tree is beautiful and yields utilty on itself. This tree also yields a stream of the same kind of fruit d_{2t}, where it happens that $d_{2t} = d_{1t} = \left(\frac{1}{2}\right) d_t \; \forall\, t$, so that the physical yields of the two kinds of trees are equal. There is one of each tree for each N individuals in the economy. Trees last forever, but the fruit is not storable. Trees are the only source of fruit.

   Each of the N individuals in the economy has preferences described by

$$E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t, s_{2t}) \tag{23}$$

   where $u(c_t, s_{2t}) = \ln c_t + \gamma \ln(s_{2t})$ where $\gamma \geq 0$, c_t denotes consumption of the fruit in period t and $s2t$ is the stock of beautiful trees owned at the beginning of the period t. The owner of a tree of either kind i at the start of the period receives the fruit d_{it} produced by the tree during that period.

   Let p_{it} be the price of a tree of type i (where $i = 1, 2$) during period t. Let R_{it} be the gross rate of returns of tree i during that period held from period t to $t+1$.

   (a) Write down the consumer optimization problem in sequential and recursive form.

   Consumer optimization problem in sequential form

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} \; E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(c_t) + \gamma \ln(s_{2t})\right) \tag{24}$$

$$\textbf{subject to} \;\; c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t}. \tag{25}$$

   Consumer optimization problem in recursive form

   The Bellman's equation is given by

$$v(d_t, s_{1t}, s_{2t}) = \max_{\{c_t, s_{1t+1}, s_{2t+1}\}} \left(\ln(c_t) + \gamma \ln(s_{2t}) + E_t \, \beta \, v(d_t, s_{1t+1}, s_{2t+1})\right) \tag{26}$$

$$\textbf{subject to} \;\; c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t}. \tag{27}$$

   (b) Define a rational expectations equilibrium.

   Defintion: A rational expectations equilibrium is an allocation $\{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{I}$, $\{\{s_{1t}^i, s_{2t}^i\}_{t=0}^{\infty}\}_{i=1}^{I}$, and a price system, $\{p_{1t}, p_{2t}\}_{t=0}^{\infty}$, such it solves each household problem and satisfies the market clear condition.

   Defintion: For the market clearing condition to exist,

   $\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} d_t^i$, $\quad \sum_{i=1}^{I} s_{1t}^i = \sum_{i=1}^{I} s_{10}^i = I, \quad \sum_{i=1}^{I} s_{2t}^i = \sum_{i=1}^{I} s_{20}^i = I,$
   where s_{10}^i and s_{20}^i are each agent's number of trees at initial time.

   The household problem is defined sequentially by each agent's utility optimization problem:

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} \; E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(c_t) + \gamma \ln(s_{2t})\right) \tag{28}$$

$$\textbf{subject to} \;\; c_t + p_{1t} s_{1t+1} + p_{2t} s_{2t+1} \leq (d_{1t} + p_{1t}) s_{1t} + (d_{2t} + p_{2t}) s_{2t} \tag{29}$$

(c) Find the pricing functions mapping the state of the ecomony at t unto p_{1t} and p_{2t} (give precise formulas). [Hint: You should be able to directly derive p_{1t} from the example seen in class, then since pricing function have to be linear you can guess a pricing function $p_{2t} = kd_t$ and solve for k parameter using Euler equation of the second stock.]

From (28), using Legrange multipilers with the given budget constraints, (29), we obtain the following:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \left((\ln(c_t) + \gamma \ln(s_{2t}) \right) \\ &+ \lambda_t((d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t} - c_t - p_{1t}s_{1t+1} - p_{2t}s_{2t+1})) \end{aligned} \tag{30}$$

First Order Conditions (30) with respect to $\{c_t, s_{1t+1}, s_{2t+1}\}$. Also using $E_0 = E_t$ since the Markov property.

$$[c_t]: \quad E_0\, \beta^t \left(\frac{1}{c_t} - \lambda_t\right) = 0 \quad \Longrightarrow \quad \frac{1}{c_t} = \lambda_t$$

$$[s_{1t+1}]: \quad E_0\, \beta^t \left(\beta\lambda_{t+1}(d_{1t+1} + p_{1t+1}) - \lambda_t p_{1t}\right) = 0 \quad \Longrightarrow \quad p_{1t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t}(d_{1t+1} + p_{1t+1})$$

$$[s_{2t+1}]: \quad E_0\, \beta^t \left(\beta\lambda_{t+1}(d_{2t+1} + p_{2t+1}) - \lambda_t p_{2t} + \beta\frac{\gamma}{s_{2t+1}}\right) = 0$$

$$\Longrightarrow p_{2t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t}(d_{2t+1} + p_{2t+1}) + \beta\frac{\gamma}{s_{2t+1}\lambda_t}$$

Combining the first order conditions above will obtain the following **pricing formula**:

$$p_{1t} = E_t\, \beta\, \frac{c_t}{c_{t+1}}(d_{1t+1} + p_{1t+1}) \tag{31}$$

$$p_{2t} = E_t\, \beta\, \frac{c_t}{c_{t+1}}(d_{2t+1} + p_{2t+1}) + \beta\frac{\gamma c_t}{s_{2t+1}} \tag{32}$$

In this economy, for general equilibrium to exist:

$$c_t = d_t \tag{33}$$

From (33), substitute into the pricing formulas in (31) and (32), obtain:

$$p_{1t} = E_t\, \beta\, \frac{d_t}{d_{t+1}}(d_{1t+1} + p_{1t+1}) \tag{34}$$

$$p_{2t} = E_t\, \beta\, \frac{d_t}{d_{t+1}}(d_{2t+1} + p_{2t+1}) + \beta\frac{\gamma d_t}{s_{2t+1}} \tag{35}$$

For p_{1t}, if assuming a the pricing function have to be of linear form obtain:

$$p_{1t} = k_{1t} d_t \tag{36}$$

From (36), substitute this into (**??**) to obtain:

$$k_{1t} d_t = E_t\, \beta\, \frac{d_t}{d_{t+1}}(d_{1t+1} + k_{1t+1} d_{t+1})$$

$$k_{1t} = E_t\, \beta\, \underbrace{k_{1t+1}}_{\Uparrow} + E_t\, \beta\, \frac{d_{1t+1}}{d_{t+1}} \tag{37}$$

$\Longrightarrow$ apply recursions on k_{1t+1}

$$k_{1t} = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_{1t+j}}{d_{t+j}} \tag{38}$$

From $d_{2t} = d_{1t} = \left(\frac{1}{2}\right) d_t \;\forall\; t$, rewrite (38) to obtain:

$$\begin{aligned} k_{1t} &= \frac{1}{2} E_t \sum_{j=1}^{\infty} \beta^j \\ &= \frac{\beta}{2(1-\beta)} \end{aligned} \tag{39}$$

For p_{2t}, if assuming a the pricing function have to be of linear form obtain:

$$p_{2t} = k_{2t} d_t \tag{40}$$

From (40), substitute this into (**??**) to obtain:

$$\begin{aligned} k_{2t} d_t &= E_t\, \beta \frac{d_t}{d_{t+1}} (d_{2t+1} + k_{2t+1} d_{t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}} \\ k_{2t} &= E_t\, \beta \underbrace{k_{2t+1}}_{\Longrightarrow \text{ apply recursions on } k_{2t+1}} + E_t\, \beta \left(\frac{d_{2t+1}}{d_{t+1}} + \frac{\gamma}{s_{2t+1}} \right) \end{aligned} \tag{41}$$

$$k_{2t} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{d_{2t+j}}{d_{t+j}} + \frac{\gamma}{s_{2t+1}} \right) \tag{42}$$

From the problem, $d_{2t} = \frac{1}{2} d_t$ and $s_{2t+1} = 1$ in equilibrium, rewrite (42) to obtain:

$$\begin{aligned} k_{2t} &= E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{1}{2} + \gamma \right) \\ &= \left(\frac{1}{2} + \gamma \right) \frac{\beta}{1-\beta} \end{aligned} \tag{43}$$

Subsititing (39) and (43) into (34) and (34), will obtain the pricing functions:

$$p_{1t} = \left(\frac{\beta}{2(1-\beta)} \right) d_t \tag{44}$$

$$p_{2t} = \left(\left(\frac{1}{2} + \gamma \right) \frac{\beta}{1-\beta} \right) d_t \tag{45}$$

(d) Prove that if $\gamma > 0$, then $R_{1t} > R_{2t} \ \forall \ t$

From (5), derive the following definitions for R_{1t} and R_{2t}.

$$R_{1t} = \frac{p_{1t+1} + d_{1t+1}}{p_{1t}} \tag{46}$$

$$R_{2t} = \frac{p_{2t+1} + d_{2t+1}}{p_{2t}} \tag{47}$$

Substiting (44) and (45) the pricing functions, into (46) and (46) can rewrite equation for R_{1t} and R_{2t} as:

$$R_{1t} = \frac{\frac{1}{2}\left(\frac{\beta}{1-\beta}\right) d_{t+1} + \frac{1}{2} d_{t+1}}{\frac{1}{2}\left(\frac{\beta}{1-\beta}\right) d_t} \tag{48}$$

$$R_{2t} = \frac{\left(\frac{1}{2}+\gamma\right)\left(\frac{\beta}{1-\beta}\right) d_{t+1} + \frac{1}{2} d_{t+1}}{\left(\frac{1}{2}+\gamma\right)\left(\frac{\beta}{1-\beta}\right) d_t} \tag{49}$$

Analyzing difference in the returns, R_{1t} and R_{2t}, is:

$$\begin{aligned} R_{1t} - R_{2t} &= \left[\frac{1}{\frac{1}{2}\frac{\beta}{1-\beta}} - \frac{1}{\left(\gamma + \frac{1}{2}\right)\frac{\beta}{1-\beta}}\right] \frac{d_{t+1}}{d_t} \\ &= \frac{1-\beta}{\beta}\left(1 - \frac{1}{1+2\gamma}\right)\frac{d_{t+1}}{d_t} \end{aligned} \tag{50}$$

From (50), if let $\gamma > 0$, $\Longrightarrow$ $R_{1t} - R_{2t} > 0$. Able to obtain:

$$R_{1t} > R_{2t} \tag{51}$$