Solving the Lucas Asset Pricing Model Using a Projection Method Approach

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Outline

1. The Lucas Asset Pricing Model
2. Legendre Polynomials and Quadrature Rules
3. The Projection Method
4. Results
5. Conclusion

The Basic Setup

Assume:

- A large number of investors.
- One stock paying stochastic dividends.
- One risk-free bond.
- All agents are identical with utility function

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma},$$

where $\gamma > 0$ is the level of risk aversion.

NOTATION

- Define the following notation:
  - c_t - the agent's consumption in time t
  - s_t, b_t - the agent's holdings of the stock and bond, respectively
  - S_t, B_t - the market holdings of the stock and bond, respectively ($S_t = 1$ and $B_t = 0$ at equilibrium)
  - p_t, q_t - the market price of the stock and bond, respectively
  - d_t - the per capita dividend paid by the stock
- For each t, the agent chooses $\{c_t, s_{t+1}, b_{t+1}\}$.
- The individual states are $z_t = \{s_t, b_t\}$ and the aggregate states are $Z_t = \{S_t, B_t, d_t\}$.

The Agent's Problem (cont)

- The agent solves

$$v(z_t, Z_t) = \max_{\{c_t, s_{t+1}, b_{t+1}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\},$$

subject to

$$c_t + p_t s_{t+1} + q_t b_{t+1} \leq s_t(p_t + d_t) + b_t \quad \forall t,$$

where s_0 and b_0 are known.

- In addition, $c_t \geq 0, \ \forall t$.

The Euler Equations

$$
\begin{aligned}
\text{Stock:} \quad & p_t c_t^{-\gamma} = \beta E_t[c_{t+1}^{-\gamma}(p_{t+1} + d_{t+1})] \\
\text{Bond:} \quad & q_t c_t^{-\gamma} = \beta E_t[c_{t+1}^{-\gamma}]
\end{aligned}
$$

or

$$
\begin{aligned}
\text{Stock:} \quad & pc^{-\gamma} = \beta E[(c')^{-\gamma}(p' + d')] \\
\text{Bond:} \quad & qc^{-\gamma} = \beta E[(c')^{-\gamma}]
\end{aligned}
$$

A TRANSFORMATION

- Suppose dividends grow according to $d_t = e^{x_t} d_{t-1}$, where $x_t = (1-\rho)\mu + \rho x_{t-1} + \varepsilon_t$ with ε_t being i.i.d. $N(0, \sigma^2)$ and $|\rho| < 1$.
- Let $v_t = \frac{p_t}{d_t}$ and $\theta = 1 - \gamma$.
- The equilibrium conditions imply that $c_t = d_t$ for all t.
- The Euler equation for the stock can be written as

$$v_t = E_t \left[\beta e^{\theta x_{t+1}} (v_{t+1} + 1) \right].$$

An Analytic Solution

Burnside's Solution

Burnside (1998) shows the exact solution for the price-dividend ratio is:

$$v_t = \sum_{j=1}^{\infty} \beta^j \exp[a_j + b_j(x_t - \mu)],$$

where

$$a_j = \theta j \mu + \frac{1}{2}\theta^2 \frac{\sigma^2}{(1-\rho)^2}\left[j - 2\frac{\rho}{1-\rho}(1-\rho^j) + \rho^2 \frac{1-\rho^{2j}}{1-\rho^2}\right]$$

and

$$b_j = \theta \frac{\rho}{1-\rho}(1-\rho^j).$$

Legendre Polynomials

Definition

The *Legendre polynomials*, P_i, can be defined recursively as

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_k(x) &= \frac{1}{k}[(2k-1)xP_{k-1} - (k-1)P_{k-2}]. \end{aligned}$$

These polynomials are orthogonal with respect to the weight function $w(x) = 1$ on $[-1, 1]$. That is,

$$\langle \varphi_i(x), \varphi_j(x) \rangle = \int_{-1}^{1} \varphi_i(x)\varphi_j(x)dx = 0, \quad \text{for all } i \neq j.$$

Gauss-Legendre Quadrature

Formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n}\omega_i f\left(\frac{(x_i+1)(b-a)}{2}+a\right)$$

- x_i and ω_i are the nodes and weights, respectively, over $[-1,1]$.
- The error is bounded by $\pi 4^{-n} M$, where

$$M = \sup_m \left[\max_{a\leq x\leq b}\frac{f^{(m)}(x)}{m!}\right].$$

- See Judd (1998) for more details.

Gauss-Hermite Quadrature

- Used to approximate integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$.
- Particularly useful for computing expectations involving normal random variables.
- x_i and ω_i are the nodes and weights, respectively, over $(-\infty, \infty)$.

Formula

Suppose $Y \sim N(\mu, \sigma^2)$. Then

$$E[f(Y)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i f(\sqrt{2}\sigma x_i + \mu)$$

The Basic Idea

Problem

Find a function f such that $v_t = f(x_t)$ and
$f(x_t) = \beta E_t \left[e^{\theta x_{t+1}} (f(x_{t+1}) + 1) \right]$, for all t.

The idea is as follows:

1. Approximate f by $\hat{f}(x) = \sum_{i=0}^{n} a_i \varphi_i(x)$, for some suitable basis.
2. Plug this into the above equation and define the residual:

$$R(x; \mathbf{a}) = \hat{f}(x) - \beta E \left[e^{\theta x'} (\hat{f}(x') + 1) \right],$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_n)$.

The Basic Idea (cont)

3. Ideally we would like to find $\mathbf{a}$ such that $R(x;\mathbf{a}) = 0$ for all x. This is generally not possible so instead we insist that it be "close" to 0 for all x.
4. Find $\mathbf{a}$ to accomplish 3.

The previous steps leave us with some questions:

- How do we choose the basis $\{\varphi_i\}_{i=0}^{n}$?
- How many of these do we use (i.e what should n be)?
- What is meant by "close to zero"?
- How will we solve for $\mathbf{a}$?

Basis Functions

We have several choices for the basis functions:

- Trigonometric functions.
  - This is a typical approach for solving PDEs but is not generally a good idea in economic problems such as this.
  - The reason is that it takes a large number of basis functions to approximate a nonperiodic function.
  - The price-dividend ratio will not be periodic.
- Monomials (i.e. $\{1, x, x^2, \dots\}$).
  - This is certainly the easiest option.
  - Will make for slow code if the degree is high.
- Legendre polynomials.
  - The orthogonality will reduce the computational burden.
  - This is the best choice.

Basis Functions (cont)

How many of these polynomials should we use in our approximation?

- It is usually not known a priori what the best value for n is.
- Usually we will solve the model many times and check the solution each time for accuracy.
- The more curvature we can expect in the true solution, the more terms we will expect to use.

Projection Conditions

- Setting up the approximation as outlined above will result in $n+1$ unknown coefficients.
- This means we need to generate $n+1$ equations to solve for the coefficients.
- There are many ways to do this (see Judd (1998) for a complete list).
- Included in the most popular methods are least squares, collocation and Galerkin.
- A detailed illustration of the Galerkin method will be given here.

The Galerkin Method

- Recall the Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_k(x) &= \frac{1}{k}[(2k-1)xP_{k-1} - (k-1)P_{k-2}]. \end{aligned}$$

- Suppose we wish to approximate the price-dividend ratio on the interval $[a, b]$.
- For $i = 0, 1, \ldots, n$ define

$$\varphi_i(x) = \begin{cases} P_i\left(2\frac{x-a}{b-a} - 1\right) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The Galerkin Method (cont)

- The φ_i are now orthogonal on $[a,b]$ with respect to the weighting function $w(x) = 1$ (in fact, they are orthogonal on any interval containing $[a,b]$). That is,

$$\langle \varphi_i(x), \varphi_j(x) \rangle = \int_a^b \varphi_i(x)\varphi_j(x)dx = 0, \quad \text{for all } i \neq j.$$

- Using this basis, form $\hat{f}(x) = \sum_{i=0}^{n} a_i \varphi_i(x)$.
- Plug $\hat{f}(x)$ into $R(x; \mathbf{a})$ and re-arrange terms.

The Galerkin Method (cont)

$$
\begin{aligned}
R(x;\mathbf{a}) &= \sum_{i=0}^{n} a_i\varphi_i(x) - \beta E_x\left[e^{\theta x'}\left(\sum_{i=0}^{n} a_i\varphi_i(x') + 1\right)\right] \\
&= \sum_{i=0}^{n} a_i\varphi_i(x) - \beta E_x\left[e^{\theta x'}\sum_{i=0}^{n} a_i\varphi_i(x') + e^{\theta x'}\right] \\
&= \sum_{i=0}^{n} a_i\varphi_i(x) - \beta E_x\left[e^{\theta x'}\sum_{i=0}^{n} a_i\varphi_i(x')\right] - \beta E_x\left[e^{\theta x'}\right] \\
&= \sum_{i=0}^{n} a_i\varphi_i(x) - \beta\sum_{i=0}^{n} a_i E_x\left[e^{\theta x'}\varphi_i(x')\right] - \beta E_x\left[e^{\theta x'}\right] \\
&= \sum_{i=0}^{n} a_i\left(\varphi_i(x) - \beta E_x\left[e^{\theta x'}\varphi_i(x')\right]\right) - \beta E_x\left[e^{\theta x'}\right]
\end{aligned}
$$

The Galerkin Method (cont)

The projection condition for the Galerkin method is that the residual must be orthogonal to each basis function, that is:

$$\langle R(x;\mathbf{a}), \varphi_k(x)\rangle = 0 \quad \text{for each } k = 0, 1, \ldots, n.$$

Imposing this gives (for each k):

$$\int_a^b \varphi_k(x) \sum_{i=0}^{n} a_i \left(\varphi_i(x) - \beta E_x\left[e^{\theta x'}\varphi_i(x')\right]\right) dx = \int_a^b \varphi_k(x)\beta E_x\left[e^{\theta x'}\right] dx,$$

or, after interchanging the (finite) sum and integral:

$$\sum_{i=0}^{n} a_i \int_a^b \varphi_k(x) \left(\varphi_i(x) - \beta E_x\left[e^{\theta x'}\varphi_i(x')\right]\right) dx = \int_a^b \varphi_k(x)\beta E_x\left[e^{\theta x'}\right] dx.$$

The Galerkin Method (cont)

The previous equation can be written more compactly as a matrix system:

$$\mathbf{Ma} = \mathbf{b},$$

where

$$M_{k,i} = \int_a^b \varphi_k(x) \left(\varphi_i(x) - \beta E_x \left[e^{\theta x'} \varphi_i(x') \right] \right) dx$$

and

$$b_k = \int_a^b \varphi_k(x) \beta E_x \left[e^{\theta x'} \right] dx.$$

Notes:

- Recall that $x' = (1-\rho)\mu + \rho x + \varepsilon'$ so that $x' \sim N(\alpha(x), \sigma^2)$. This allows the expectations to be computed using Gauss-Hermite.
- The remaining integrals are computed using Gauss-Legendre.

Error Analysis

- The following norms were used to compute the error:

$$E_1 = 100 \times \frac{1}{N} \sum_{t=1}^{N} \left| \frac{y_t - \hat{y}_t}{y_t} \right| \quad \text{and} \quad E_\infty = 100 \times \max \left\{ \left| \frac{y_t - \hat{y}_t}{y_t} \right| \right\},$$

  where y_t is the true solution and $\hat{y}_t$ is the approximated solution.
- The code was run for increasing orders until each of the above errors seemed to roughly converge. In most cases this was around 10^{-7}.
- $N = 20$ was used for the error computations.

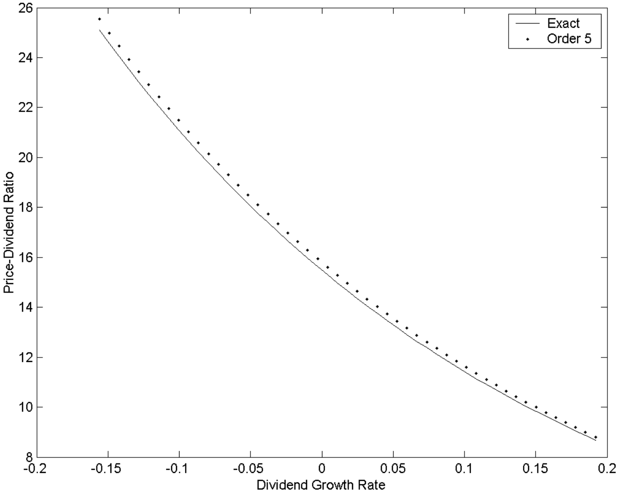
Error Analysis (cont)

- *Benchmark Values:* $\mu = 0.0179$, $\sigma = 0.0348$, $\rho = -0.139$, $\theta = -1.5$, $\beta = 0.95$
- Large values of ρ (near 1) and large negative values of θ will result in the most curvature in the exact solution.
- Burnside's solution converges if and only if the following is satisfied:

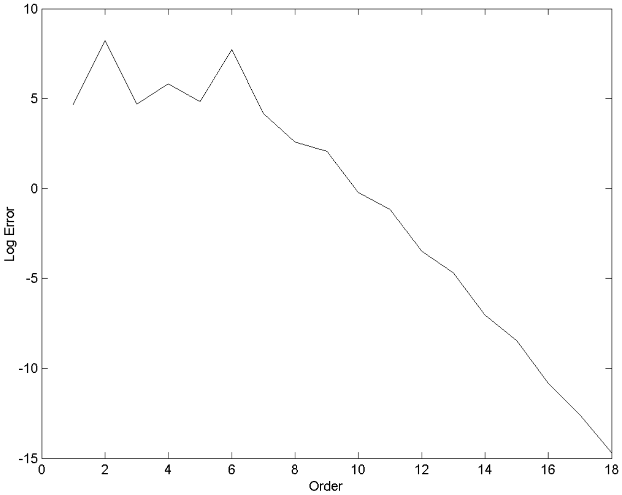
$$\beta \exp\left[\theta\mu + \frac{1}{2}\theta^2 \frac{\sigma^2}{(1-\rho)^2}\right] < 1.$$

- With all other parameters held at their benchmark levels, ρ cannot exceed 0.85 if the true solution is to converge.

Order 5 Plot for $\rho = 0.7$



Log Error Plot for $\rho = 0.8$



Pros and Cons of the Method

- Pros
  - Easy to code for a general n.
  - Reasonably fast to compute even for high orders.
  - The *idea* works for multiple dimensions also.
- Cons
  - Computationally challenging to implement in many dimensions (number of unknown coefficients grows exponentially).
  - Not as straightforward to use on a nonlinear problem.

The End

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