1. Consider the standard cake eating problem with a modification of the transition equation for the cake

$$W' = RW - c \tag{1}$$

if $R > 1$. the cake yields a positive return, whereas if $R\epsilon(0,1)$ the cake depreciates.

(a) If preferences are of the form $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$, calculate the optimal sequence of consumption. What happens when $\sigma = 1$? and when $\sigma = 0$?

To solve the constrained optimization problem, need to find the sequence of $\{c_t\}_1^T$ $\{W_t\}_2^{T+1}$ that satisifies

$$V_t(W\,') = \max \sum_{t=1}^{T} \beta^{(t-1)}\; u(c_t) \tag{2}$$

subject to the transition equation (1), which holds for $t = 1, 2, 3, \ldots, T$. Also, subject to nonegative constrants on comsuming the cake given by $c_t \geq 0$ and $W_{t+1} \geq 0$

Adding period 0 consumption, the solution to (2) can be rewitten as

$$V_{t+1}(W) = \max_{\{c_t\}_t^T}\{u(c_t) + \beta V_t(W\,')\} \tag{3}$$

By using the Legrange multipliers, combining the first order conditions we obtain the **Euler equation** that is necessary condition of optimality for any t.

$$u\,'(c_t) = R\,\beta\, u\,'(c_{t+1}) \tag{4}$$

By letting $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}, \Rightarrow u\,'(c) = c^{-\sigma}$, subsitute in (4) obtain the following:

$$(c_t)^{-\sigma} = R\,\beta\,(c_{t+1})^{-\sigma} \tag{5}$$

$$c_{t+1} = (R\,\beta)^{\frac{1}{\sigma}}\, c_t \tag{6}$$

Let $\tilde{\beta} = (R\,\beta)^{\frac{1}{\sigma}}$, and using properites of geometric sequence then (6) can be expressed as consumpution of period $T+1$ in terms of period 0.

$$c_{t+1} = (\tilde{\beta})^T\, c_0 \tag{7}$$

Using the sequence of transition equations from (1), multiply each equation by R^{T-i} where $i = 0, 1, \ldots, T$.

$$\begin{array}{ll}
\text{at } t=0 & R^T\, W^1 = R_{T+1}\, W_0 - R_T\, c_0 \\
\text{at } t=1 & R^{T-1}\, W_2 = R^T\, W_1 - R^{T-1}\, c_1 \\
\text{at } t=2 & R^{T-2}\, W_3 = R^{T-1}\, W_0 - R^{T-2}\, c_2 \\
\vdots & \vdots \\
\text{at } t=T & R^0\, W_{T+1} = R^0\, W_T - R^0\, c_T
\end{array}$$

When sum all the transaction equations obtain the following:

$$R^0\, W_{T+1} = R^{T+1}\, W_0 - \sum_{t=0}^{T} R^{T-t}\, c_t \tag{8}$$

Let consumption at period $W_{T+1} = 0$ and intial cake size is known, $W_0 = \overline{W}_0$. By subsituting (7) into (8) for c_i, obtain the following expression for c_0.

$$R^{T+1}\,\overline{W}_0 = \sum_{t=0}^{T} R^{T-t}\,(\tilde{\beta})^t\,c_0$$

$$R^{T+1}\,\overline{W}_0 = R^T\,c_0 \sum_{t=0}^{T} R^{-t}\,(\tilde{\beta})^t$$

$$R\,\overline{W}_0 = c_0 \sum_{t=0}^{T} \left(\frac{\tilde{\beta}}{R}\right)^t$$

$$c_0 = \frac{R\,\overline{W}_0}{\sum_{t=0}^{T} \left(\frac{\tilde{\beta}}{R}\right)^t} \tag{9}$$

From (7), subsititing (9), obtain a solution for the consumption at any period $t = 1, 2, \ldots, T$ on the finite horizon.

$$c_t = \frac{\tilde{\beta}^t\,R\,\overline{W}_0}{\sum_{t=0}^{T} \left(\frac{\tilde{\beta}}{R}\right)^t} \tag{10}$$

Recall that $\tilde{\beta} = (R\,\beta)^{\frac{1}{\sigma}}$.

$$c_t = \frac{(R\,\beta)^{\frac{t}{\sigma}}\;R\,\overline{W}_0}{\sum_{t=0}^{T} \left(\frac{(R\,\beta)^{\frac{1}{\sigma}}}{R}\right)^t} \tag{11}$$

When $\sigma = 1$ obtain:

$$c_t = \frac{(R\,\beta)^t\;R\,\overline{W}_0}{\sum_{t=0}^{T}\;(\beta)^t} \tag{12}$$

When $\sigma = 0$ obtain:

$$u(c) = c \Rightarrow u'(c) = 1$$

From (4), obtain the following:

$$1 = R\,\beta \tag{13}$$

(b) Show that we can reduce the cake eating problem to a system of nonlinear equations of the form $F(W) = 0$, the optimal sequence of consumption, $\{c_t{}^*\}_1^T$.

Note: If find the sequence $\{W_t{}^*\}_1^{T+1}$, can find sequence of consumption by using (1).

Using (3), subject to the constaints (1), **Note:** $c_t = RW_T - W_{t+1}$, consumption at period $W_{T+2} = 0$ and intial cake size is known, $W_0 = \overline{W}_0$, subsituting in (4), obtain the following system of nonlinear equations with t equations and t unknowns:

$$F(\mathbf{W}) = \begin{bmatrix} u'(RW_0 - W_1) = R\beta u'(RW_1 - W_2) \\ u'(RW_1 - W_2) = R\beta u'(RW_2 - W_3) \\ u'(RW_2 - W_3) = R\beta u'(RW_3 - W_4) \\ \vdots \\ u'(RW_T - W_{T+1}) = R\beta u'(RW_{T+1} - W_{T+2}) \end{bmatrix}$$

(c)

2. Consider the previous version of the cake eating problem with a law of motion of the form,

$$W' = y + \delta - c\,, \qquad \text{where } \delta \,\epsilon\, (0, R) \tag{14}$$

(a) Formulate the Bellman equation of the infinite horizon problem.

As in the case of the previous problem, calculate the optimal sequences, $\{c_t\}_1^\infty$ $\{W_t\}_2^\infty$, that solves the following equation, subject to the constraint in (14).

$$V(W) = \max \sum_{t=0}^{T} \beta^t \; u(c_t) \tag{15}$$

Equation (15) can be rewritten as, subject to (14), where $0 \leq W' \leq W$:

$$V(W) = \max\{\, u(y + \delta\, W - W') + \beta\, V(W')\,\} \tag{16}$$

(b) If preferences are of the form $u(c) = \ln(c)$ and $y = 0$, calculate the optimal value function and the implied decision rules. Does the size of δ matter for the optimal values?

$$u(c) = \ln(c) \Rightarrow u'(c) = \frac{1}{c} \quad \text{and} \quad y = 0$$

$$W' = \delta\, W - c \quad \Rightarrow c = \delta\, W - W' \tag{17}$$

Using the following form below for the value function, derive the first order conditions to find the optimal policy functions.

$$V(W) = \alpha_0 + \alpha_1 \ln(W) \qquad \text{with} \Rightarrow V'(W) = \frac{\alpha_1}{W} \tag{18}$$

First Order Conditions (16)

$$[W']: \qquad u'(c) = \beta\, V'(W')$$

$$\frac{1}{\delta\, W - W'} = \beta \left(\frac{\alpha_1}{W'}\right) \qquad \textbf{subsituting } (17) \textbf{ and } (18).$$

$$W' = \beta\, \alpha_1\, \delta\, W - \alpha_1\, \beta\, W'$$

Solve for W' to obtain the **optimal policy function for savings**:

$$W' = \left(\frac{\beta\, \alpha_1 \delta}{1 + \beta\, \alpha_1}\right) W \tag{19}$$

Subsituting (19) in (17) to obtain the **optimal policy function for consumption**:

$$c = \delta\, W - \left(\frac{\beta\, \alpha_1 \delta}{1 + \beta\, \alpha_1}\right) W = \left(\frac{\delta + \alpha_1\, \delta\, \beta - \alpha_1\, \delta\, \beta}{1 + \alpha_1\, \beta}\right) W$$

$$c = \left(\frac{\delta}{1 + \beta\, \alpha_1}\right) W \tag{20}$$

Need to solve for the coefficents of the value function in (18). Setting (18) equal to (16), and solve for coefficients of (18) $\{\alpha_0,\, \alpha_1\}$. This will be done by subsituting (20) and (19) into (16).
Solving for $\{\alpha_0,\, \alpha_1\}$ and subsituting into (20) and (19) will obtain the optimal value function.

$$\alpha_0 + \alpha_1\, \ln(W) = u(c) + \beta^t\, V(W') \qquad \textbf{from } (18) \textbf{ and } (16)$$

$$\alpha_0 + \alpha_1\, \ln(W) = \ln(c) + \beta\,(\alpha_0 + \alpha_1\, \ln(W')) \qquad \textbf{from given } u(c) = \ln(c) \textbf{ and } (18)$$

$$\alpha_0 + \alpha_1\, \ln(W) = \ln\left(\frac{\delta\, W}{1+\beta\,\alpha_1}\right) + \beta\,\alpha_0 + \beta\,\alpha_1\, \ln\left(\frac{\beta\,\alpha_1 \delta\, W}{1+\beta\,\alpha_1}\right) \qquad \textbf{from } (20) \textbf{ and } (19)$$

$$\alpha_0 + \alpha_1\, \ln(W) = \underbrace{\ln\left(\frac{\delta}{1+\beta\,\alpha_1}\right) + \beta\,\alpha_0 + \beta\,\alpha_1\, \ln\left(\frac{\beta\,\alpha_1 \delta}{1+\beta\,\alpha_1}\right)}_{constant = \alpha_0} + \underbrace{(1+\beta\,\alpha_1)}_{\alpha_1} \ln(W)$$

Therefore, able to setup the following equations and solve for $\{\alpha_0,\, \alpha_1\}$.

$$\alpha_1 = 1 + \beta\,\alpha_1$$

$$\alpha_1 = \frac{1}{1-\beta} \tag{21}$$

$$\alpha_0 = \ln\left(\frac{\delta}{1+\beta\,\alpha_1}\right) + \beta\,\alpha_0 + \beta\,\alpha_1\, \ln\left(\frac{\beta\,\alpha_1 \delta}{1+\beta\,\alpha_1}\right)$$

$$\alpha_0 = \ln\left(\frac{\delta}{1+\dfrac{\beta}{1-\beta}}\right) + \beta\,\alpha_0 + \frac{\beta}{1-\beta}\, \ln\left(\frac{\dfrac{\beta\,\delta}{1-\beta}}{1+\dfrac{\beta}{1-\beta}}\right) \qquad \textbf{Subsituting } (21)$$

$$\alpha_0(1-\beta) = \ln\left(\delta\,(1-\beta)\right) + \frac{\beta}{1-\beta}\, \ln(\beta\,\delta)$$

$$\alpha_0 = \frac{1}{1-\beta} \ln\left(\delta\,(1-\beta)\right) + \frac{\beta}{(1-\beta)^2}\, \ln(\beta\,\delta) \tag{22}$$

Using (22) and (21), able to obtain the **optimal policy function** and the **optimal value function**

$$c = \left(\frac{\delta}{1 + \alpha_1 \beta}\right) W \qquad \textbf{from } (20)$$

$$c = \left(\frac{\delta}{1 + \dfrac{\beta}{1 - \beta}}\right) W \qquad \textbf{subsituting } (21)$$

$$c = \delta(1 - \beta)W \tag{23}$$

$$W' = \delta W - C \qquad \textbf{from } (17)$$

$$W' = \delta \beta W \tag{24}$$

Therefore, (24) and (23) is the **optimal policy function**. Using the solutions for $\{\alpha_0, \alpha_1\}$ from (21) and (22) able to obtain the following **optimal value function**

$$V(w) = \frac{1}{1 - \beta} \ln(\delta(1 - \beta)) + \frac{\beta}{(1 - \beta)^2} \ln(\beta \delta) + \frac{1}{1 - \beta} \ln(W) \tag{25}$$

Does

(c)

3. Consider the instanteous utility function that depends on previous consumption $u(c_{t-1}, c_t)$. Previous consumption affects present consumption, so consumers have to tak into account that today's consumption choice will also affect future utility besides the size of the cake.

   (a.) Solve the T-period cake eating problem using this preferences.