Chapter 2

Asset Prices and Consumption

2.1 Fundamental Equation: Stochastic Discount Rate and Payoffs

The goal of this section is to determine the value of an uncertain monetary flow of resources at time t, prior to the resolution of uncertainty. The asset return $R_{t+1.}$ will be treated as a random variable, so we have to decide how much are we willing to pay to hold this asset. The consumer solves

$$\max_{\{a\}} u(c_t) + \beta E_t[u(c_{t+1})]$$

$$s.to \qquad c_t + p_t a = \omega_t$$

$$c_{t+1} = \omega_{t+1} + R_{t+1} a$$

where $\omega_{t,}$and ω_{t+1} denote the resource endowment at each period , and a is a portfolio with price p_t, and the uncertain future return is R_{t+1} next period. We can rewrite the problem as

$$\max_{\{a\}} u(\omega_t - p_t a) + \beta E_t[u(\omega_{t+1} + R_{t+1} a)]$$

the first-order conditions for an interior solution are

$$-u'(\omega_t - p_t a)p_t + \beta E_t[u'(\omega_{t+1} + R_{t+1} a)R_{t+1}] = 0,$$

rearranging terms

$$p_t u'(c_t) = \beta E_t[u'(c_{t+1})R_{t+1}]$$

where $p_t u'(c_t)$ is the loss of utility associated to buy an additional unit of the asset measured in terms present consumption, and $\beta E_t[u(c_{t+1})R_{t+1}]$ is the expected discounted utility received for holding the asset at $t+1$. Alternatively, we can write the expression as the price that a consumer is willing to pay for an asset with expected return R_{t+1} and consumption flow (c_t, c_{t+1}):

$$p_t = E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}\right)$$

We can rewrite the asset pricing equation as follows

$$p_t = E_t[m_{t+1}x_{t+1}]$$

where $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$, represents the **stochastic discount factor** and represents the asset payoff $x_{t+1} = R_{t+1}$. In the absence of uncertainty, the asset price depends on the discounted value of payments. Formally,

$$p_t = \frac{x_{t+1}}{R^f} = \frac{x_{t+1}}{1 + r_{t+1}}$$

where $R^f > 1$, represents the discount rate or the return of a riskless assets. Risky assets have a lower price than riskless asset because they are adjusted by the risk premium.

$$p_t^i = \frac{1}{R^i} E[x_{t+1}^i]$$

The risk premium factor is asset specific, and it depends on the correlation of the asset return and future consumption. The general structure of the asset pricing equation is useful to analyze assets of different forms, depending on the definition of p_t, and x_{t+1}.

Price and Payoff Notation

	Price p_t	Payoff x_{t+1}
Stock	p_t	$p_{t+1} + d_{t+1}$
Return	1	R_{t+1}
Price-dividend ratio	p_t/d_t	$(p_{t+1}/d_{t+1} + 1)\, d_{t+1}/d_t$
Excess return	0	$R_{t+1}^e = R_{t+1}^a - R_{t+1}^f$
One-period bond	p_t	1
Risk-free rate	1	R^f
Option	C	$max(S_T - K, 0)$

2.2 Applications of the Fundamental Pricing Equation

2.2.1 Risk-Free Rate

We consider the presence of a risk-less asset with positive return $R^f_{t+1} \equiv x_{t+1}$, and $p_{t+1} = 1$. Using the fundamental equation

$$1 = E_t[m_{t+1} R^f_{t+1}]$$

with an uncertain return

$$1 = E_t[m_{t+1}] R^f_{t+1}$$

however consumption still uncertain, therefore isolating R^f_{t+1}, we can rewrite the expression as

$$R^f_{t+1} = \frac{1}{E_t[m_{t+1}]}$$

or

$$R^f_{t+1} = \frac{1}{E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}]}$$

If the equality hold, consumers are indifferent between buying and selling assets because there are no arbitrage opportunities. If this is not the case, two different events can occur:

$$R^f_{t+1} > \frac{1}{E_t[m_{t+1}]}$$

If the price is larger than the consumption flow, the consumers should be willing to sell liquidity (save), and consume more at $t+1$. Alternatively,

$$R^f_{t+1} < \frac{1}{E_t[m_{t+1}]}$$

Consumers are borrowing to consume more at t, then, they will repay next period.

Example 1: Certain consumption and isoelastic preferences

We consider a utility function of the form

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

where marginal utility is $u(c_t) = c_t^{-\gamma}$, and relative risk aversion is $R_R(c) = \gamma$. The parameter γ, represents the relative risk aversion and is also the inverse of the intertemporal elasticity of substitution. The implied expression is given by

$$R^f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

or

$$R^f = \frac{(1+g_c)^{\gamma}}{\beta}$$

where $1 + g_c = c_{t+1}/c_t$ denotes consumption growth. The interest rate can be high because of three reasons.

1. Consumers are impatient i.e. β is low. If every body is impatient and wants to consume in the present it takes a high interest rate to convince them to consume.

2. When consumption growth is high. That comes from the fact that consumers are saving to consume in the future taking advantage of high returns.

3. Preference for smoothing is high, then it takes a high rate of return to convince consumers to deviate from a smooth consumption path.

Example 2: Uncertain consumption

If consumption growth is lognormal, then we can rewrite the fundamental equation using the following notation

$$r^f_{t+1} = \ln R^f_{t+1} \rightarrow e^{r^f_{t+1}} = R^f_{t+1}$$

$$\beta = e^{-\delta} = \frac{1}{e^{\delta}}, \quad \text{and} \quad \delta > 1$$

$$\triangle \ln c_{t+1} = \ln c_{t+1} - \ln c_t$$

to derive the equation we have

$$R_{t+1}^{f} = \frac{1}{E_t[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}]} = \frac{1}{E_t[e^{-\delta}e^{\ln}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}]}$$

$$R_{t+1}^{f} = \frac{1}{E_t[e^{-\delta}e^{-\gamma\cdot\triangle\ln c_{t+1}}]}$$

lognormal growth implies

$$E[e^z] = e^{E[z]+\frac{\sigma^2}{2}(z)}$$

then,

$$e^{r_{t+1}^{f}} = \frac{1}{e^{-\delta}e^{-\gamma\cdot E_t\triangle\ln c_{t+1}+\gamma^2\frac{\sigma^2}{2}(\triangle\ln c_{t+1})}}$$

rearranging terms

$$e^{r_{t+1}^{f}} = e^{\delta}e^{\gamma\cdot E_t\triangle\ln c_{t+1}-\gamma^2\frac{\sigma^2}{2}(\triangle\ln c_{t+1})}$$

taking log

$$r_{t+1}^{f} = \delta + \gamma\cdot E_t\triangle\ln c_{t+1} - \gamma^2\frac{\sigma^2}{2}(\triangle\ln c_{t+1}).$$

Next, we can analyze the relationship between the risk-free rate and the parameters of the equation

- $\triangle\delta$ (degree of patience)$\rightarrow\bigtriangledown\beta\rightarrow\triangle r_{t+1}^{f}$: The argument is the same as above. If all the individuals are very impatience, a higher risk free rate is necessary to induce them to save.
- $\triangle\ln c_{t+1}$ (higher consumption growth)$\rightarrow$ $\triangle r_{t+1}^{f}$: If consumption is growing is because the interest rate is high , that is $\bigtriangledown c_t\rightarrow\triangle a_{t+1}\rightarrow\triangle c_{t+1}\rightarrow\triangle\ln c_{t+1}$. As a result high returns, r_{t+1}^{f} reduce consumption and increase consumption growth.
- γ (risk aversion or the inverse of the intertemporal elasticity of substitution): The argument is the same as before.
- $\sigma^2\cdot\triangle\ln c_{t+1}$ (Consumption volatility): When consumption is volatile, risk averse consumers respond by saving. Therefore, an increase in volatility induces saving and reduces the risk-free rate necessary to force agents to save.

2.2.2 Risk Premium

So far we have assumed no correlation between the payoff and the stochastic discount rate. If both variables are correlated, then we know that

$$E(m \cdot x) = E(m) \cdot E(x) + cov(m, x),$$

or

$$cov(m, x) = E(m \cdot x) - E(m) \cdot E(x)$$

We can redefine the pricing equation as

$$p = E(m) \cdot E(x) + cov(m, x)$$

using the expression of the risk-free rate from the previous section $R^f = \frac{1}{E[m]}$:

$$p = \underbrace{\frac{E(x)}{R^f}} + \underbrace{cov(m, x)}.$$

The price of a risky asset depends on two terms. The first term is the discounted value of future payoff. In the absence of uncertainty $E(x) = x$, so the asset price is $p = \frac{x}{R^f}$. This is the price in the presence of risk neutral agents, and it occurs when either $u(c_t) = A$, or $u(c_t) = c_t$. In this case, $cov(m, x) = 0$. The second term is the risk premium used to correct the price. Then we have that when $cov(m, x) > 0 \rightarrow \triangle p$, otherwise $cov(m, x) < 0 \rightarrow \bigtriangledown p$.

To understand the risk correction it is useful to substitute its value on the pricing equation

$$p_t = \frac{E_t(x_{t+1})}{R^f_{t+1}} + \underbrace{cov(\beta \frac{u'(c_{t+1})}{u'(c_t)}, x)}$$

Notice that $\triangle c_t \rightarrow \bigtriangledown u'(c_{t+1})$, and viceversa.

- If $cov(c_{t+1}, x) > 0 \rightarrow cov(u'(c_{t+1}), x) < 0 \rightarrow \bigtriangledown p$

This asset is positively correlated with consumption, and it give high returns when consumption growth and low returns otherwise increasing consumption volatility. Consumers require a higher return or a lower price to hold this asset. Formally,

$$p_t < \frac{E_t(m_{t+1})}{R^f_{t+1}}$$

- If $cov(c_{t+1}, x) < 0 \rightarrow cov(u'(c_{t+1}), x) > 0 \rightarrow \triangle p$

  This asset return is negatively correlated with consumption, and allows consumption smoothing. Consumers value this asset and as a result its price is higher and its returns is lower. Insurance contracts are a classic example. The payoff when consumption is low, and cost money when consumption is high

$$p_t > \frac{E_t(m_{t+1})}{R^f_{t+1}}$$

We can do the same analysis with returns. The fundamental pricing equation is given by

$$1 = E(m \cdot R)$$

decomposing the covariance

$$1 = E(m) \cdot E(R) + cov(m, R)$$

$$E(R) = \frac{1}{E(m)} - \frac{cov(m, R)}{E(m)}$$

If there exists a risk-less asset we have

$$E(R) = R^f - \frac{cov(m, R)}{E(m)}$$

Substituting m

$$\overline{R} = R^f - \frac{cov[u'(c_{t+1}), R]}{E[u'(c_{t+1})]}$$

- Again assets with positive covariance with consumption, $cov(c, R) > 0$, make consumption more volatile, $cov(u'(c), R) < 0$, therefore the expected return required to hold them $\overline{R}$ need to be higher.

- Assets with negative covariance with consumption, $cov(c, R) < 0$, provide insurance and make consumption less volatile, $cov(u'(c), R) > 0$, therefore the required return to hold them is lower.

2.2.3 Idiosyncratic Risk

It is direct to see that idiosyncratic risk does not affect prices. When the return of an asset is not correlated with the stochastic discount factor, that is $cov(m, x) = 0$, then, the asset does not receive a risk compensation. Consequently,

$$p = E(m) \cdot E(x)$$

or

$$p = \frac{E(x)}{R^f}$$

2.3 Risk Aversion and the Equity Premium

Consider two different consumption alternatives

$$\left\langle \begin{array}{l} c - \pi \text{ with certainty} \\ \left\langle \begin{array}{ll} c - y & \text{with prob=0.5} \\ c + y & \text{with prob=0.5} \end{array} \right. \end{array} \right.$$

we want to find the certainty equivalent that satisfies

$$u(c - \pi) = 0.5u(c - y) + 0.5u(c + y)$$

we need to solve a non-linear function $\pi(c, y)$ that satisfies

$$u(c - \pi(c, y)) = 0.5u(c - y) + 0.5u(c + y)$$

We can use a first-order Taylor expansion series around $x_0 = c$ in both sides. Formally

$$u(c - \pi) = u(c) + u'(c)(c - \pi - c) = u(c) - \pi u'(c)$$

and

$$\begin{aligned} 0.5[u(c - y) + u(c + y)] &= 0.5[u(c) + u'(c)(c + y - c) + 0.5u''(c)(c + y - c)^2 \\ &\quad ... + u(c) + u'(c)(c - y - c) + 0.5u''(c)(c - y - c)^2] \\ &= 0.5[2u(c) + \frac{1}{2}(yu'(c) - yu'(c)) + y^2 u''(c)] \\ &= 0.5[2u(c) + y^2 u''(c)] \end{aligned}$$

Combining both expressions we obtain

$$u(c) - \pi u'(c) = 0.5[2u(c) + y^2 u''(c)]$$
$$\pi u'(c) = 0.5 y^2 u''(c)$$

or

$$\frac{\pi}{y} = \frac{1}{2} y \left(- \frac{u''(c)}{u'(c)} \right)$$

for a CRRA utility function $u(c) = c^{1-\gamma}(1-\gamma)^{-1}$ we have

$$\frac{\pi}{y} = \frac{1}{2} y \left(\frac{y}{c} \right)$$

the percentage of the premium with respect to the size of the loss. That changes linearly with risk aversion.

$$\frac{\partial(\frac{\pi}{y})}{\partial \gamma} = \frac{1}{2} y$$