Chapter 3

General Equilibrium under Uncertainty

3.1 Environment

3.2 Arrow-Debreu Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s)\}_{s=1}^S\}_{i=1}^I$ and state contingent prices $p = \{p_0, \{p_1(s)\}_{s=1}^S\}$ such that

1. Consumer i takes prices p as given and chooses the allocation x^i that solves

$$\max_{\{c_0^i, c_1^i(s)\}_{s=1}^S} U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s)),$$

$$s.t. \qquad p_0 c_0^i + \sum_{s=1}^S p_1(s) c_1^i(s) \leq p_0 y_0^i + \sum_{s=1}^S p_1(s) y_1^i(s),$$
$$c_0^i, c_1^i(s) \geq 0$$

2. Markets clear

$$\sum_{i=1}^I c_0^i \leq \sum_{i=1}^I y_0^i = Y_0,$$
$$\sum_{i=1}^I c_1^i(s) \leq \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s,$$

Example 1 (Equilibrium prices): Consider a two-state economy where consumers have preferences of the form $u(c) = \ln c$. The consumer problem becomes

$$\max_{\{c_1^i(s)\}_{s=1}^S} \sum_{s=1}^S \pi_s \log(c_1^i(s)),$$

$$s.t. \qquad \sum_{s=1}^S p_1(s) c_1^i(s) \leq \sum_{s=1}^S p_1(s) y_1^i(s), \quad c_1^i(s) \geq 0$$

The first-order conditions of the consumer problem are given by

$$\frac{\pi_s}{c_1^i(s)} = \lambda p_1(s),$$

that implies an equal expected consumption expenditure across states. Formally,

$$\pi_s p_1(j) c_1^i(j) = \pi_j p_1(s) c_1^i(s)$$

Summing across all agents we have

$$\pi_s p_1(j)[c_1^1(j) + ... + c_1^I(j)] = \pi_j p_1(s)[c_1^1(s) + ... + c_1^I(s)]$$

or

$$\pi_s p_1(j) Y(j) = \pi_j p_1(s) Y(s)$$

Rearranging terms

$$\frac{p_1(j)}{p_1(s)} = \frac{\pi_j}{\pi_s} \frac{Y(s)}{Y(j)}$$

Example 2 (Idiosyncratic Uncertainty): Assume that the aggregate endowments are constant across states, formally, $Y(s) = Y(j)$ for s and j. Then, the equilibrium prices are given by

$$p_1(s) = \pi_s$$

Substituting in the consumers FOC we obtain perfect risk-sharing

$$c_1^i(s) = c_1^i(j)$$

for all i and s. From the households budget constraint we can compute the equilibrium consumption given by

$$c_1^i(s) = \sum_{s=1}^S \pi_s y_1^i(s),$$

Example 3 (Aggregate Uncertainty): Assume that the aggregate endowments fluctuate across states. In particular, consider an stochastic growth rate of the form $Y(s) = g(s)Y(1)$, where $g(s) > 0$. Then, the relative prices between a given state s and the normalized state 1 is

$$\frac{p_1(j)}{p_1(1)} = \frac{\pi_j}{\pi_1}\frac{g(s)Y(1)}{Y(1)},$$

or

$$\frac{p_1(j)}{p_1(1)} = \frac{\pi_j}{\pi_1}g(s)$$

3.3 Sequential Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s), b_1^i(s)\}_{s=1}^S\}_{i=1}^I$ and state contingent prices $q = \{q_1(s)\}_{s=1}^S$ such that

1. Consumer i takes prices q as given and chooses the allocation x^i that solves

$$\max_{\{c_0^i, \{c_1^i(s), b_1^i(s)\}_{s=1}^S\}} U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s)),$$

$$\begin{aligned} s.t. \qquad c_0^i &= y_0^i - \sum_{s=1}^S q_1(s) b_1^i(s), \\ c_1^i(s) &\leq y_1^i(s) + b_1^i(s), \qquad \forall s, \\ c_0^i, c_1^i(s) &\geq 0 \end{aligned}$$

2. Markets clear

$$\begin{aligned} \sum_{i=1}^I b_1^i(s) &\leq 0, \qquad \forall s, \\ \sum_{i=1}^I c_0^i &\leq \sum_{i=1}^I y_0^i = Y_0, \\ \sum_{i=1}^I c_1^i(s) &\leq \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s, \end{aligned}$$

Example 4 (Equilibrium prices): Consider the same example of the previous section where we ignore time 0 consumption, but not the asset markets. The consumer problem becomes

$$\max_{\{c_1^i(s), b_1^i(s)\}_{s=1}^S} \sum_{s=1}^S \pi_s \log(c_1^i(s)),$$

$$\begin{aligned} s.t. \qquad \sum_{s=1}^S q_1(s) b_1^i(s) &= y_0^i, \\ c_1^i(s) &\leq y_0^i + b_1^i(s) \qquad \forall s, \end{aligned}$$

Let λ and $\mu(s)$ denote the Lagrange multipliers of the budget constraints. The first-order conditions of the consumer problem are given by

$$\begin{aligned} \frac{\pi_s}{c_1^i(s)} &= \mu^i(s), \\ q_1(s)\lambda^i &= \mu^i(s) \end{aligned}$$

Rearranging terms we obtain an equal expected consumption expenditure across states. Formally,

$$\pi_s q_1(j) c_1^i(j) = \pi_j q_1(s) c_1^i(s)$$

Summing across all agents we have

$$\pi_s q_1(j)[c_1^1(j) + ... + c_1^I(j)] = \pi_j q_1(s)[c_1^1(s) + ... + c_1^I(s)]$$

or

$$\frac{q_1(j)}{q_1(s)} = \frac{\pi_j}{\pi_s} \frac{Y(s)}{Y(j)}$$

Substituting into the FOC we obtain

$$\frac{Y(1)}{c_1^i(1)} = ... = \frac{Y(S)}{c_1^i(S)}$$

According to this expression, individual consumption is perfectly correlated with output, and aggregate consumption. In particular, if $\triangle Y(s)$ then $\triangle c_1^i(s)$ changes in the same magnitude, but is imperfectly correlated with individual income. The expected individual income determines the levels, but not the ratios across states of nature.

Example 5 (Idiosyncratic Uncertainty): We can use the previous example to compute the implied portfolios. Consider a specialized example with only two shocks. In the second period,

$$c_1^i(1) = c_1^i(2)$$

Now, we can compute the demand for Arrow securities combining time 0 and time 1 budget constraints,

$$q_1(1)b_1^i(1) + q_1(2)b_1^i(2) = 0$$
$$y_1^i(1) + b_1^i(1) = y_1^i(2) + b_1^i(2)$$

Rearranging terms we obtain,

$$\widehat{b}_1^i(1) = (1-\pi)[y^i(2) - y^i(1)]$$
$$\widehat{b}_1^i(2) = \pi[y^i(1) - y^i(2)]$$

3.4 Financial Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s)\}_{s=1}^S, \{a_{1j}^i\}_{j=1}^J\}_{i=1}^I$ and state contingent prices $Q = \{Q_{1j}\}_{j=1}^J$ such that

1. Consumer i takes prices Q as given and chooses the allocation x^i that solves

$$\max_{x^i} U(c_0^i) + \beta \textstyle\sum_{s=1}^S \pi_s U(c_1^i(s)),$$

$$\begin{aligned} s.t. \qquad c_0^i &= y_0^i - \textstyle\sum_{j=1}^J Q_{1j} a_{1j}^i, \\ c_1^i(s) - y_1^i(s) &\leq r_{11}(s)a_{11}^i(s) + ... + r_{1J}(s)a_{1J}^i(s), \qquad \forall s, \\ c_0^i, c_1^i(s) &\geq 0 \end{aligned}$$

2. Markets clear

$$\textstyle\sum_{i=1}^I a_{1j}^i(s) \leq 0, \qquad \forall j$$
$$\textstyle\sum_{i=1}^I c_0^i \leq \textstyle\sum_{i=1}^I y_0^i = Y_0,$$
$$\textstyle\sum_{i=1}^I c_1^i(s) \leq \textstyle\sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s,$$

It is direct to show that all these economies are equivalent with complete markets. In particular, the sequential market structure is a special case of the financial markets economy, where the return of the asset in all the states but one are zero. The equilibrium prices have to satisfy the no-arbitrage property, so we can use the equilibrium allocations from the other economies to compute the price of any asset.

3.5 Pareto Efficiency

The social planner problem is the easiest way to find the optimal consumption allocations. Then, we can use the second welfare theorem to compute the implied equilibrium prices. Formally, the social planner solves

$$\max_x \sum_{i=1}^I \lambda_i \left[U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s)) \right],$$

$$\begin{aligned} s.t. \qquad \sum_{i=1}^I c_0^i &= \sum_{i=1}^I y_0^i = Y_0, \\ \sum_{i=1}^I c_1^i(s) &= \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s \end{aligned}$$

where $\sum_{i=1}^I \lambda_i = 1$. Let α be the Lagrange multiplier of the time 0 resource constraint, and $\mu(s)$ the multiplier of the resource constraint at time 1 in state s. The optimal allocation satisfies

$$\lambda_i U'(c_0^i) = \alpha$$
$$\lambda_i \beta \pi_s U'(c_1^i(s)) = \mu(s)$$

together with the $s+1$ resource constraints. Combining time 0 first-order conditions we have

$$\lambda_1 U'(c_0^1) = ... = \lambda_I U'(c_0^I)$$

In a symmetric equilibrium, individual consumption is a constant fraction of aggregate output. At time 1 we have a similar result.

$$\lambda_1 \beta \pi_s U'(c_1^1(s)) = ... = \lambda_I \beta \pi_s U'(c_1^I(s)),$$

or

$$U'(c_1^1(s)) = ... = U'(c_1^I(s))$$

The optimal allocation implies perfect insurance within the state. Individual consumption is only correlated with aggregate consumption, but not individual income shocks. That is

$$c_0^i = \frac{1}{I} Y_0,$$
$$c_1^i(s) = \frac{1}{I} Y_1(s)$$

Higher aggregate shocks lead to higher individual consumption levels, but the distribution does not matter. The determinate for asset market prices are not consumption levels, but ratios. Therefore, the ratio of consumption or marginal utility does not usually depend on the relative weight that the planner assigns.