Chapter 4

General Equilibrium with Complete Markets

4.1 Environment

- Finite number of states $s \in S$
- $\pi(s'/s) = prob(s_{t+1} = s'/s_t = s)$ is a first-order Markov chain

  $\pi_0(s) = prob(s_0 = s)$ is the initial distribution

  $\pi(s^t)$ is a sequence of probability measures to achieve a particular history

  $$s^t = (s_t, s_{t-1}, s_{t-2}, ..., s_1, s_0)$$

  This probability can be computed via recursion

  $$\pi(s^t) = \pi(s_t/s_{t-1})\pi(s_{t-1}/s_{t-2})...\pi(s_1/s_0)\pi(s_0)$$

  This is the unconditional probability when s_0 has not been observed yet. When s_0 has been observed, we then have the conditional probability

  $$\pi(s^t/s_0) = \pi(s_t/s_{t-1})\pi(s_{t-1}/s_{t-2})...\pi(s_1/s_0)$$

  where $\pi(s^t) = \pi(s^t/s_0)\pi(s_0)$
- Finite number of agents $i \in I$
- Endowment for each household $y_t^i = y^i(s_t)$ is a time-invariant function that only depends on the the shock at time t.

- Endowments are publicly observable
- An allocation for agent i is defined as state contingent function $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$
- Preferences are represented by

$$U(c^i) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

or

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s^t))$$

where the utility function $u(\cdot)$ satisfies $u' > 0$, $u'' < 0$, C^2 and the Inada conditions $\lim_{t \to 0} u'(c) = +\infty$.

- An allocation is a list of sequence of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for all i. An allocation is said to be feasible if it satisfies

$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

Notice that consumption can depend on history, but the period income only depends on the realization of the shock.

4.2 Arrow-Debreu Markets

Household trade dated state-contingent claims to consumption. There is a complete set of claims. Trade takes place at $t = 0$ after the shock has been realized. The price of a claim on time t consumption contingent on history s^t is denoted by $p_t^0(s^t)$. The superscript 0 refers to the date at which trades occur, while the time subscript t refers to the date that deliveries are to be made. A price system is a sequence of functions $\{p_t^0(s^t)\}_{t=0}^{\infty}$.

A given household i solves

$$U(c^i) = \max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s^t))$$

$$s.to \quad \sum_{t=0}^{\infty}\sum_{s^t} p_t^0(s^t)c_t^i(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} p_t^0(s^t)y^i(s_t)$$
$$c_t^i(s^t) \geq 0$$

The single budget constraint implicitly assumes complete markets because it allows unrestricted trade in all states of nature.

Definition (Competitive Equilibrium): A competitive equilibrium is a feasible allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^\infty\}_{i=1}^I$ and a price system $\{p_t^0(s^t)\}_{t=0}^\infty$ such that the allocation solves each household problem.

Proposition: *The competitive equilibrium allocation is not history dependent.*

$$c_t^i(s^t) = c^i(s^t)$$

Proof: The first-order conditions of the consumer problem are given by

$$\beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \gamma^i p_t^0(s^t)$$

For two different consumers that face the same prices we have

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\gamma^i}{\gamma^j} \quad \forall t, s$$

The ratios of marginal utilities between pairs of agents is constant across time and states. In general, that will not imply constant consumption levels, but proportional. Latter we will show that in absence of aggregate uncertainty, consumption will be constant across time and states of nature.

The relative consumption is given by

$$c_t^i(s^t) = u'\left(u'(c_t^j(s^t))\frac{\gamma^i}{\gamma^j}\right)^{-1}$$

This fact comes from combining the first-order of the consumer problem with the resource constraint

$$\sum_{i=1}^{I} u'\left(u'(c_t^j(s^t))\frac{\gamma^i}{\gamma^j}\right)^{-1} = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

If the right-hand side does not depend on history, it only depends on the existing shock s_t. Therefore, the left-hand side does not depend on history either.■

The equilibrium price function is derived from the consumer first-order conditions

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{\gamma^i}$$

At $t = 0$, we also have

$$p_0^0(s^0) = \frac{u'(c_0^i(s^0))}{\gamma^i}$$

or

$$p_t^0(s^t) = \beta\pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))}$$

where $p_0^0(s^0) = 1$. The ratio of expected marginal utilities gives the stochastic discount factor, and the return of the state-contingent claim is one unit of consumption. Therefore, the price has to be lower than one. Once we determine the consumption allocation, we can compute the equilibrium prices.

4.2.1 Risk Sharing

Economist are interested on the insurance properties of financial markets, and increase welfare. Consider a utility function of the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where $\sigma > 0$. The optimality condition of the market equilibrium implies

$$\frac{(c_t^i)^{-\sigma}}{(c_t^j)^{-\sigma}} = \frac{\gamma^i}{\gamma^j} \quad \forall t$$

or

$$c_t^j = c_t^i \left(\frac{\gamma^i}{\gamma^j}\right)^{\frac{1}{\sigma}} \quad \forall t$$

Complete markets assumption implies that consumption allocations to distinct agents are constant fractions of another. With this preferences, individual consumption is perfectly correlated with aggregate output or consumption, but is not correlated with individual income $y^i(s_t)$. The fraction of consumption that each agent receives is independent of s^t. Hence, the model exhibits an extensive cross-state and cross-time consumption smoothing.

4.2.2 No Aggregate Uncertainty

We consider an economy with two types of consumers, and a continuum of each type. The Markov process s_t takes place on the unit interval $s_t \in [0,1]$, such that $y^1(s_t) = s$ and $y^2(s_t) = 1-s$. In the absence of aggregate uncertainty, we know that the optimal choice implies perfect insurance $c_t^i = c_0^i$,

$$\beta^t \pi(s^t/s_0) \frac{u'(c_t^1(s^t))}{u'(c_0^1(s^0))} = p_t^0(s^t) = \beta\pi(s^t/s_0)\frac{u'(c_t^2(s^t))}{u'(c_0^2(s^0))}$$

That is in equilibrium, we have

$$\frac{u'(c_t^1(s^t))}{u'(c_0^1(s^0))} = \frac{u^2(c_t^1(s^t))}{u^2(c_0^1(s^0))}$$

From the first-order conditions of the consumer problem, we have

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{\gamma^i}$$

Substituting the first-order condition into the budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)\frac{u'(c_0^i)}{\gamma^i}[c_0^i - y^i(s_t)] = 0$$

$$\frac{u'(c_0^i)}{\gamma^i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)[c_0^i - y^i(s_t)] = 0$$

given that $u'(c_0^i)/\gamma^i \neq 0$, then it must be the case that

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)[c_0^i - y^i(s_t)] = 0$$

or

$$c_0^i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi(s^t/s_0) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)y^i(s_t)$$

where $\sum_{s^t} \pi(s^t/s_0) = 1$, so we have

$$c_0^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) y^i(s_t)$$

Finally, we check feasibility

$$\begin{aligned} c_0^1 + c_0^2 &= (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) y^1(s_t) + (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) y^2(s_t) \\ &= (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) [\underbrace{y^1(s_t)}_{s} + \underbrace{y^2(s_t)}_{1-s}] = (1-\beta) \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t/s_0) = 1 \end{aligned}$$

Using the optimal consumption levels, we can compute the implicit asset prices.

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{u'(c_t^i(s^t))}{\gamma^i}$$

with constant marginal utility, and using the usual normalization $\gamma^i = u'(c_t^i(s^t))$ and $p_0^0(s^0) = 1$. We obtain

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)$$

where remember that $\pi(s^t/s_0)$ is the conditional probability for this particular history when s_0 has been observed. An important feature is that prices do not depend on the idiosyncratic income shock. It only depends on the particular realization of a given history.

We can further specialize the example assuming a particular endowment process for both consumers. Formally, assume that $y^1 = (1, 0, 1, 0, ...)$ and $y^2 = (0, 1, 0, 1, ...)$. In this case $p_t^0(s^t) = \beta^t$. The implied consumption allocations for both consumers are given by

$$c_0^1 = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) y^i(s_t) = (1-\beta) \sum_{t=0}^{\infty} \beta^{2t} 1$$

or

$$c_0^1 = \frac{(1-\beta)}{(1-\beta^2)} = \frac{(1-\beta)}{(1-\beta)(1+\beta)} = \frac{1}{1-\beta}$$

and for the other consumer we have,

$$c_0^2 = \frac{\beta}{1-\beta}$$

The first-consumer is relatively wealthier because it receives the high shock on the first-period. That allows high to consume more because the present value of his/her future income is higher.

4.3 Contingent Claims or Sequential Markets Structure

In a seminal paper Arrow (1964) showed that one-period securities are enough to implement complete markets, as long as a new one-period market re-opens for trading next period. In this economy, trade takes place at each date and state $t \geq 0$ using a set of contingent claims to one-period ahead state consumption. We prove that with a full array of these one period set of claims, the sequential market structure attains the same consumption allocation as the competitive equilibrium with Arrow-Debreu market structure.

In this economy, the sequential budget constraint is given by

$$c_t^i(s_t) + \sum_{s^{t+1}} Q(s_{t+1}/s_t) b_{t+1}^i(s_{t+1}) = y_t^i(s_t) + b_t^i(s_t) \quad \forall s$$

where $Q(s_{t+1}/s_t)$ denotes the price of one unit of consumption t time $t+1$ contingent on state s_{t+1} given that today is state s_t. We assume that this function does not depend on t. Notice that consumption only depends on the existing shock s_t, and does not depend on history. All the history for household i is summarized by its present wealth given by $b_t^i(s_t)$.

A given household i solves

Definition: A sequential equilibrium is an allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s_t), b_{t+1}^i(s_{t+1})\}_{t=0}^\infty\}_{i=1}^I$ and a price system $\{Q(s_{t+1}/s_t)\}_{t=0}^\infty$ such that

i) the allocation solves each household problem, and

$$U(c^i) = \max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s_t))$$

$$s.to \quad c_t^i(s_t) + \sum_{s^{t+1}} Q(s_{t+1}/s_t) b_{t+1}^i(s_{t+1}) = y_t^i(s_t) + b_t^i(s_t) \quad \forall s$$

$$c_t^i(s_t) \geq 0 \quad b_{t+1}^i(s_{t+1}) \geq -B(s_t) \quad \forall s$$

ii) Markets clear

$$\sum_{i=1}^{I} c_t^i(s_t) = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

$$\sum_{i=1}^{I} b_{t+1}^i(s_{t+1}) = 0$$

Proposition: If $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^I$ is the solution of the Arrow-Debreu competitive equilibrium, this allocation also is the solution of the sequential equilibrium.

Proof: From the first-order conditions of the sequential problem we have

$$Q(s_{t+1}/s_t) = \beta\pi(s_t/s_t) \frac{u'(c_{t+1}^i(s_{t+1}))}{u'(c_t^i(s_t))}$$

together with a transversality condition

$$\lim_{t\to\infty} \sum_{s^{t+1}} Q(s_{t+1}/s_t) b_{t+1}^i(s_{t+1}) = 0$$

That implies $b_{t+1}^i(s_{t+1}) > 0$ if $Q(s_{t+1}/s_t) = 0$, or $b_{t+1}^i(s^{t+1}) = 0$ if $Q(s_{t+1}/s_t) > 0$. The first-order conditions of the Arrow-Debreu equilibrium are

$$\frac{p_{t+1}^0(s^{t+1})}{p_t^0(s^t)} = \beta\pi(s^t/s_0) \frac{u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))}$$

That implies

$$Q(s_{t+1}/s_t) = \frac{p_{t+1}^0(s^{t+1})}{p_t^0(s^t)}$$

Finally, we need to show that the Arrow-Debreu consumption allocation satisfies the sequential budget constraint. In this case, we choose the initial level of wealth so the allocations are the same $b_0^i = 0$ for all i. Then, the portfolio decisions should be the same in both economies. To show it we need to add up all the budget constraints across states of nature s and across time t, basically across all histories s^t. We start at time $t = 0$

$$\begin{array}{lll} t=0 & s=1 & p_0^0(1)[c_0^i(1) - y_0^i(1)] = p_0^0(1)b_0^i(1) - \sum_{s^1} p_0^0(1)Q(s_1/1)b_1^i(1) \\ \dots & \dots & \dots \\ t=1 & s=S & p_0^0(S)[c_0^i(S) - y_0^i(S)] = p_0^0(S)b_0^i(S) - \sum_{s^1} p_0^0(S)Q(s_1/S)b_1^i(S) \end{array}$$

If we add them up we have

$$\sum_{s^0} p_0^0(s^0)[c_0^i(s^0) - y_0^i(s_0)] = \sum_{s^0} p_0^0(s^0)b_0^i(s_0) - \sum_{s^1} \underbrace{p_0^0(s^1)Q(s_1/s_0)}_{p_1^0(s^1)} b_1^i(s_1)$$

For the next periods we have

$$\sum_{s^1} p_1^0(s^1)[c_1^i(s^1) - y_1^i(s_1)] = \sum_{s^1} p_1^0(s^1)b_1^i(s_1) - \sum_{s^2} p_2^0(s^2)b_2^i(s_2)$$

$$\dots$$

$$\sum_{s^{t-1}} p_{t-1}^0(s^{t-1})[c_{t-1}^i(s^{t-1}) - y_{t-1}^i(s_{t-1})] = \sum_{s^{t-1}} p_{t-1}^0(s^{t-1})b_{t-1}^i(s_{t-1}) - \sum_{s^t} p_t^0(s^t)b_t^i(s_t)$$

$$\sum_{s^t} p_t^0(s^t)[c_t^i(s^t) - y_t^i(s_t)] = \sum_{s^t} p_t^0(s^t)b_t^i(s_t) - \sum_{s^{t+1}} p_{t+1}^0(s^{t+1})b_{t+1}^i(s_{t+1})$$

If we add them all up,

$$\sum_{s^0} p_0^0(s^0)[c_0^i(s^0) - y_0^i(s_0)] + \dots + \sum_{s^t} p_t^0(s^t)[c_t^i(s^t) - y_t^i(s_t)] = \sum_{s^{t+1}} p_{t+1}^0(s^{t+1})b_{t+1}^i(s_{t+1})$$

if we take the limit in both sides de have

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t)[c_t^i(s^t) - y_t^i(s_t)] = \lim_{t\to\infty} \sum_{s^{t+1}} p_{t+1}^0(s^{t+1})b_{t+1}^i(s_{t+1}) = 0$$

4.4 Pareto Efficient Allocations

It is useful to have a welfare measure to compare the outcomes of different trading mechanism. We focus on Pareto efficient.

Definition (Pareto Efficiency): An allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s_t)\}_{t=0}^\infty\}_{i=1}^I$ is said to be Pareto efficient, if there not exists another feasible allocation $\{\widetilde{c}^i\}_{i=1}^I$ such that

$$\begin{aligned} U(\widetilde{c}^i) &\geq U(c^i) \qquad \forall i \\ U(\widetilde{c}^i) &> U(c^i) \qquad \text{some } i \end{aligned}$$

The set of Pareto efficient allocation can be calculated by computing the so called social planner problem. Consider a social planner that has to allocate resources among a large number of households. We assume that each consumer receives a time invariant discount rate $\lambda^i \in (0,1)$, and $\sum_{i=1}^I \lambda^i = 1$. The benevolent planner maximizes

$$U(c^1, ..., c^I) = \max_{\{c_t^i(s^t)\}_{i=1}^I} \sum_{i=1}^I \lambda^i \sum_{t=0}^\infty \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s^t))$$

$$\begin{aligned} s.to \quad \sum_{i=1}^I c_t^i(s^t) &= \sum_{i=1}^I y^i(s_t) = Y(s_t) \\ c_t^i(s^t) &\geq 0 \end{aligned}$$

Let μ denote the Lagrange multiplier of the resource constraint. The first-order conditions for a given consumer i with respect to $c_t^i(s^t)$ are

$$\lambda^i \beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \mu$$

Notice that marginal utility of consumption only depends on the aggregate variables, not on the individual income shock $y^i(s_t)$. Formally,

$$u'(c_t^i(s^t)) = \frac{\mu}{\lambda^i \beta^t \pi(s^t/s_0)}$$

or

$$c_t^i(s^t) = u\left(\frac{\mu}{\lambda^i \beta^t \pi(s^t/s_0)}\right)^{-1}.$$

For two different consumers i and j we have

$$\frac{\lambda^i \beta^t \pi(s^t/s_0) u'(c_t^i(s^t))}{\lambda^j \beta^t \pi(s^t/s_0) u'(c_t^j(s^t))} = 1$$

or
$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\lambda^j}{\lambda^i}$$
Clearly, the allocation of consumption across households depends on the relative weight that the social planner assigns to each household. In particular, if $\lambda^i > \lambda^j$ then $u'(c_t^i(s^t)) < u'(c_t^j(s^t))$, and $c_t^i(s^t) > c_t^j(s^t)$. The agent with higher weight receives more consumption. In a symmetric allocation $\lambda^i = \lambda^j$ all agents receive the same allocation, $c_t^i(s^t) = \alpha Y(s_t)$, where $\alpha = 1/I$. Individual consumption only depends on the aggregate shock, not on the idiosyncratic labor income shock. Finally, we can replace the optima consumption levels on the first-order conditions

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{u'(\alpha Y(s_t))}{u'(\alpha Y(s_t))} = \frac{\lambda^j}{\lambda^i}$$

and obtain $\lambda^j = \lambda^i$ that both agents need to have the same initial wealth to achieve the symmetric allocation. If all agents do not have the same initial wealth, it is necessary to implement lump-sum taxes to achieve this allocation.

4.5 First and Second Welfare Theorems

First, we want to prove the so called first-welfare theorem. The theorem highlights some of the nice welfare properties of complete markets economies.

Proposition (First-welfare theorem): An equilibrium allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^\infty\}_{i=1}^I$ in the market economy is Pareto efficient.

Proof: Suppose the contrary, then there exists another feasible allocation $\{\tilde{c}^i\}_{i=1}^I$ that Pareto dominates the equilibrium allocation. At the equilibrium prices $\{p_t^0(s^t)\}_{t=0}^\infty$, this allocation has to cost strictly more than the endowment for the individual that can be improved. Otherwise this agent is not maximizing utility. That is
$$\sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t)\tilde{c}_t^i(s^t) > \sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) y^i(s_t)$$

for the other consumers this constraint is satisfied with equality. If we add up all the constraints we find

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t^0(s^t)\sum_{i=1}^{I}\widetilde{c}_t^i(s^t) > \sum_{t=0}^{\infty}\sum_{s^t} p_t^0(s^t)\sum_{i=1}^{I} y^i(s_t)$$

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t^0(s^t)\left(\sum_{i=1}^{I}\widetilde{c}_t^i(s^t) - Y(s_t)\right) > 0$$

given that $p_t^0(s^t) > 0$ for all t and s, the alternative allocation $\{\widetilde{c}^i\}_{i=1}^I$ is not feasible. That contradicts the assumption of Pareto efficient allocations. Clearly, there exists better allocations but there are not feasible.■

Proposition (Second-welfare theorem): An allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^I$ is Pareto efficient, there exists a price system that supports this allocation as a market equilibrium.

Proof: If we compare the first-order conditions of the social planner

$$\lambda^i \beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \mu$$

with the competitive equilibrium from the previous section

$$\beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \gamma^i p_t^0(s^t)$$

It is clear, that both economies will deliver the same allocations if $\mu/\lambda^i = \gamma^i p_t^0(s^t)$. There exist a vector of relative weight $\{\lambda^i\}_{i=1}^I$, such that the solution of both economies is the same given the initial distribution of entitlements. In particular, we can use the social planner allocations two compute the optimal consumption, and the implied equilibrium price system.

For the symmetric case, that is $\lambda^1 = ... = \lambda^I$, where $c_t^i(s^t) = \alpha Y(s_t)$ for all i and $\alpha = 1/I$ represents the individual share on aggregate output.

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)\frac{u'(\alpha Y(s_t))}{u'(\alpha Y(s_0))}$$

The price of state-contingent claims depends on the co-movements of output.■

Assume that the changes in aggregate output across time and states is given by $Y(s_t) = g(s_t)Y(s_0)$, where you can think of $Y(s_0)$ as the average level of output. We can rewrite this equation as

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{u'(\alpha g(s_t)Y(s_0))}{u'(\alpha Y(s_0))}$$

In the absence of aggregate uncertainty $g(s_t) = 1$ for all s and t. Then, the equilibrium prices are given by

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)$$

We obtain the same pricing that with risk-neutral preferences $u(c) = c$. In the presence of aggregate uncertainty and isolastic preferences $u'(c) = c^{-\gamma}$,

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{(\alpha g(s_t)Y(s_0))^{-\gamma}}{(\alpha Y(s_0))^{-\gamma}}$$

or

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) g(s_t)^{-\gamma}$$

The price of consumption goods is lower in states with high output growth, and higher in states with low output growth. Agents with high endowments in periods with low output are relatively wealthier.

One way to test the model is to use estimate a process for consumption growth, and see whether the implied equilibrium prices satisfy the some properties observed in the data.

The advantage of the second welfare theorem, is that we can use the social planner problem to compute the optimal allocations, and the used them to derive the equilibrium prices. Notice that the equilibrium prices do not depend on the social planner weight, because they depend on the ratio of marginal utilities, and this ratio is unaffected by the weight. We will exploit this result to solve Lucas model of asset prices.

4.6 Lucas Model of Asset Prices

The two previous specifications do not specify the market structure that yields a constant interest rate for example. Lucas asset pricing model uses a simple exchange economy to determine the pricing function. The economy considers a large number of identical agents which receive no labor income. We consider and economy populated by a large number of identical households solving

$$\max_{\{c_t, s_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.a. \qquad c_t + p_t s_{t+1} = s_t(p_t + d_t)$$
$$s_{t+1} \geq -B$$

where B is a large positive constant that never binds but prevents Ponzi schemes. Notice that we have set $y_t = 0$ in all t. The only durable good is a set of "trees" which are equal in number to the number of people in the economy. At each period t, each tree yields a fruit or dividend in the amount d_t to its owner. We assume that the dividend is nonstorable, but the tree is perfectly durable. The solution of this problem yields

$$p_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}(p_{t+1} + d_{t+1})]$$

together with a transversality condition $\lim_{j \to \infty} \beta^j u'(c_{t+j}) p_{t+j} = 0$. This condition says that in the limit consumer will not hold assets if the price is positive, or will hold positive amounts if the price is zero.

The competitive equilibrium consumption allocation of this economy can be readily be computed once we notice that the economy can be treated as autarkic. Because preference and endowment patterns are the same across individuals, there can be no gains from trade. In equilibrium it must be the case that $c_t = d_t$ because the utility function $u(\cdot)$ is strictly increasing (that means no satiation), and the dividend is the only source of consumption goods. We can deal with a representative consumer directly.

In equilibrium, prices have to be such that markets clear. That means that the total amount of borrowing in the economy is zero, and the share holdings has to be one, $s_t = 1$. Substituting the equilibrium conditions in the Euler equation, and using the law of iterated expectations we conclude that the price of a share must satisfy

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j} + E_t \lim_{j\to\infty} \beta^j u'(d_{t+j}) p_{t+j}$$

The transversality condition of the consumer problem rules out solutions that include a bubble term. If the last term where positive $E_t \lim_{j\to\infty} \beta^j u'(d_{t+j})p_{t+j} > 0$, the marginal utility of selling shares excess the marginal utility of holding assets and consume the expected flow $p_t u'(d_t) > E_t \sum_{j=1}^{\infty} \beta^j u'(d_{t+j})d_{t+j}$. Consequently, all households would sell share to increase their consumption, and as a result the price of a share will fall. We have a similar argument if the additional term is negative. There in equilibrium it must be the case that

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}$$

or

$$p_t = E_t \sum_{j=1}^{\infty} m_{t+j} d_{t+j}$$

where $m_{t+j} = \beta^j u'(d_{t+j})/u'(d_t)$ represents the stochastic discount factor. This equation is a generalization of the random walk theory of stock prices, in which the share price is an expected discounted stream of dividends but with a time-varing and stochastic discount rate m_{t+j} that is different from one as in the previous case. We can decompose the price of the asset in two terms: the discounted value of the consumption flow and the correction term for risk. Formally,

$$p_t = \sum_{j=1}^{\infty} \left(E_t(m_{t+j}) E_t(d_{t+j}) + cov(m_{t+j}, d_{t+j}) \right)$$

with risk neutral preferences we have that $m_{t+j} = \beta^i$, and with perfect insurance we have that $u'(c_t) = E_t u'(c_{t+1})$ and $cov(m_{t+j}, d_{t+j}) =$

0. So the price of an asset is the discounted sum of future dividends

$$p_t = \sum_{j=1}^{\infty} E_t(d_{t+j}),$$

In general, that will not be the case and the asset will be adjusted by the premium factor. Given that $c_t = d_t$, it must be the case that when there is a good shock $\triangle d_{t+j} \rightarrow \triangle c_{t+j} \rightarrow \nabla u'(c_{t+j}) \rightarrow \nabla m_{t+j} = u'(c_{t+j})/u'(c_t)$. Then, $cov(m_{t+j}, d_{t+j}) < 0$ so we have

$$p_t < \sum_{j=1}^{\infty} E_t(m_{t+j}) E_t(d_{t+j})$$

if we normalize $E_t(m_{t+j}) = 1$, we have that the price of a risky asset should be lower than the expected discounted stream of its dividends. That also means that the return of that asset is higher because otherwise households will not buy this asset.

This version of the Lucas model has been used to generate allocations and price of assets, and compare them with the data. These asset pricing models can be constructed as follows:

1. We describe the preferences, technology and endowments. Given a particular market structure where agents are allowed to buy and sell assets, we solve for the equilibrium consumption allocations.

2. Sometimes there exists a planning problem whose solution equals the competitive equilibrium. Therefore, we can equate the consumption that appears on the Euler equation, and compute the implied asset price at time t as a function of the state of the economy at t.

In our endowment economy, a benevolent social planner would solve

$$\max_{\{c_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} s.to \qquad & c_t \leq d_t, \\ & c_t \geq 0 \end{aligned}$$

After substituting the consumption allocations into the pricing equations we derive the standard equation for a price of a share

$$u'(d_t)p_t = E_t[\beta u'(d_{t+1})(p_{t+1} + d_{t+1})]$$

or

$$p_t = E_t[\beta m_{t+1} x_{t+1}]$$

where $m_{t+1} = u'(c_{t+1})/u'(c_t)$ and $x_{t+1} = (p_{t+1} + d_{t+1})$. Next, we want to study some special cases

Example 1: Logarithmic utility function

Consider a utility function of the form $u(c_t) = \ln c_t$, where $u'(c_t) = c_t^{-1}$. If we replace this expression in the pricing equation we obtain

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_t}{d_{t+j}} d_{t+j}$$

rearranging terms

$$p_t = d_t E_t \sum_{j=1}^{\infty} \beta^j$$

or

$$p_t = \frac{1}{1-\beta} d_t$$

This equation is an example of an asset-pricing function that maps the state of the economy at t, d_t into the price of a capital asset at t. In particular, the dividend at time t is all the information required to predict the price. In this particular case $m_{t+j} = d_t/d_{t+j}$ and does not necessarily need to be one. The price is a linear function of the aggregate state of the economy. This is a property that we will exploit in detail in this chapter.

Example 2: Risk neutrality or perfect insurance

If the utility function is linear in consumption, $u(c_t) = c_t$, then the ratio of marginal utilities is constant. That is $m_{t+j} = u'(c_{t+j})/u'(c_t) = 1$. Then, the price of a share at time t is

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t d_{t+j}$$

in this case the price of the share depends on the stochastic properties of the dividend process.

- **First-order autoregressive process:** If we assume that dividends follow a first-order autoregressive process

$$d_{t+1} = \alpha + \rho d_t + \varepsilon_{t+1}$$

where ε_{t+1} is white noise, that is $E(\varepsilon_{t+1}) = 0$. Then, the expected value of the dividend is at $t+1$ is

$$\begin{aligned}
E_t[d_{t+1}] &= \alpha + \rho d_t \\
E_t[d_{t+2}] &= E[\alpha + \rho d_{t+1} + \varepsilon_{t+2}] = \alpha + \rho E[d_{t+1}] = \alpha + \rho(\alpha + \rho d_t) \\
E_t[d_{t+3}] &= \alpha + \rho E[d_{t+2}] = \alpha(1 + \rho + \rho^2) + \rho^3 d_t \\
\vdots & \qquad\qquad \vdots \\
E_t[d_{t+k}] &= \alpha(1 + \rho + \rho^2 + \ldots + \rho^{k-1}) + \rho^k d_t
\end{aligned}$$

or

$$p_t = \sum_{j=1}^{\infty} \beta^j [\alpha(1 + \rho + \rho^2 + \ldots + \rho^{j-1}) + \rho^j d_t]$$

$$p_t = \alpha \sum_{j=1}^{\infty} \beta^j \sum_{j=1}^{\infty} \rho^j + \sum_{j=1}^{\infty} (\beta\rho)^j d_t$$

- **I.i.d. shocks:** If we assume that the process is independent and identically distributed according to $\varepsilon \sim N(0, \sigma^2)$, then

$$d_t = \alpha + \varepsilon_{t+1}$$

the price of the dividend flow is given by

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t(\alpha + \varepsilon_{t+1}) = \frac{\alpha}{1-\beta},$$

In this case prices are set to the mean value of the dividend process.

4.6.1 Equivalent Allocations

Next, we show that the Lucas model, or asset structure yields the same allocations that the Arrow-Debreu markets

Proposition: If $\{c^i\}_{i=1}^I = \{\{c_t^i(z_t), s_{t+1}^i(z_t)\}_{t=0}^\infty\}_{i=1}^I$ is the solution of the Lucas model, then, the consumption and asset allocation also is the solution of the Arrow-Debreu competitive equilibrium.

Proof: We start with the sequential Lucas constraint for a particular realization of the dividend shock.

$$c_t - y_t = s_t(q_t + d_t) - q_t s_{t+1}$$

we define the Arrow-Debreu prices as

$$\frac{p_t}{p_{t+1}} = \frac{q_{t+1} + d_{t+1}}{q_t}$$

or $p_t q_t = p_{t+1}(q_{t+1} + d_{t+1})$. Now we multiply each sequential budget constraint by is respective price $p_t, p_{t+1}, p_{t+2}, ...$ Formally, we have

$$\begin{aligned} p_t[c_t - y_t] &= p_t(q_t + d_t)s_t - p_t q_t s_{t+1} \\ p_{t+1}[c_{t+1} - y_{t+1}] &= p_{t+1}(q_{t+1} + d_{t+1})s_{t+1} - p_{t+1} q_{t+1} s_{t+2} \\ p_{t+2}[c_{t+2} - y_{t+2}] &= p_{t+2}(q_{t+2} + d_{t+2})s_{t+2} - p_{t+2} q_{t+2} s_{t+3} \\ &... \end{aligned}$$

If we add them up

$$\sum_{t=1}^{\infty} p_t[c_t - y_t] = p_t(q_t + d_t)s_t - s_{t+1}[\underbrace{p_t q_t - p_{t+1}(q_{t+1} + d_{t+1})}_{=0}] - s_{t+2}[\underbrace{p_{t+1} q_{t+1} - p_{t+2}(q_{t+2} + d_{t+2})}_{=0}] + ...$$

Now we need to solve for $p_0(q_0 + d_0)s_0$

$$p_0 q_0 s_0 + p_0 d_0 s_0$$

where $p_0 = p_1(q_1 + d_1)/q_0$

$$\frac{p_1(q_1 + d_1)}{q_0} q_0 s_0 + p_0 d_0 s_0 = p_1(q_1 + d_1)s_0 + p_0 d_0 s_0 = p_1 q_1 s_0 + (p_1 d_1 + p_0 d_0)s_0$$

that is

$$\sum_{t=1}^{\infty} p_t d_t s_0$$

Combining all together we have

$$\sum_{t=1}^{\infty} p_t [c_t - y_t] = \sum_{t=1}^{\infty} p_t d_t s_0$$

Now, we just need to add-across states of nature

$$\sum_{t=1}^{\infty} \sum_{s^t} p_t^0(s^t)[c_t(s^t) - y_t(s_t)] = \sum_{t=1}^{\infty} \sum_{s^t} p_t^0(s^t) d_t(s_t) s_0$$

The model is equivalent to the Arrow-Debreu complete markets model, where agents receive an endowment or initial share on the tree, s_0. The price of the shares can be used to price the equivalent state $t-0$ contingent claims.

4.6.2 The Random Walk Theory of Consumption

The next two theories emerge from studying marginal conditions for the consumer's problem and imposing some restrictions upon them. As we will see latter on, it is possible to describe simple market equilibrium setups that deliver these restrictions.

First, we analyze the random walk theory of consumption formulated by Hall (1978). According to Hall the evolution of future consumption follows a random walk, and no variable in the information set can be used to predict it.[1] This theory is based on the stochastic Euler equation derived in the previous section. Formally,

$$u'(c_t) = \beta E_t[u'(c_{t+1}) R_{t+1}]$$

Hall assumes that in the economy there exists a risk-free rate asses with constant return $R_t = R > 1$. Under this assumption

[1] Put in prespectivr this theory

$$C_t = f(Y_t)$$

and discuss the PIH in constrast with standard Keynesian theory.

we can rewrite the Euler equation as

$$u'(c_t) = \beta E_t[u'(c_{t+1})]R$$

or

$$E_t[u'(c_{t+1})] = (\beta R)^{-1} u'(c_t)$$

This equation shows that the marginal utility of consumption follows a univariate first-order Markov process and that no other variables in the information set help to predict. We can rewrite the previous expression to include an error term on it. Formally,

$$E_t[u'(c_{t+1})] = (\beta R)^{-1} u'(c_t) + \varepsilon_{t+1}$$

We can further specialize the problem if we assume some particular preferences.

Example 1: Quadratic utility function

Consider a simple quadratic utility function given by

$$u(c_t) = a + bc_t + dc_t^2,$$

where a, b, and d are constants. The first and second derivatives are

$$u'(c_t) = b + 2dc_t$$
$$u''(c_t) = 2d$$

where we need to assume that $d < 0$ and (????). Substituting the expression in the Euler equation we have

$$E_t[b + 2dc_{t+1}] = (\beta R)^{-1}(b + 2dc_t) + \varepsilon_{t+1}$$

Assuming that $(\beta R)^{-1} = 1$, we obtain,

$$2dE_t[c_{t+1}] = 2dc_t + \varepsilon_{t+1}$$

where expected consumption is,

$$E_t[c_{t+1}] = c_t + \delta_{t+1}$$

where $\delta_{t+1} = \frac{\varepsilon_{t+1}}{2d}$. If tomorrow marginal utility on consumption c_{t+1} follows an stochastic process and no other variable on the information set can help to predict expected marginal utility, then the evolution of future consumption follows a random walk.

Example 2: Constant relative risk aversion utility function

Next, we consider a constant relative risk aversion utility function,

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},$$

the equation that needs to be tested is given by,

$$E_t[c_{t+1}^{-\gamma}] = (\beta R)^{-1} c_t^{-\gamma}$$

using the same set of assumptions, $(\beta R)^{-1} = 1$, we obtain a different solution. In particular,

$$E_t[c_{t+1}] = c_t$$

This problem satisfies the certainty equivalence result. The solution to the problem with uncertainty is equivalent to the solution the problem under certainty. Formally,

$$c_{t+1}^{-\gamma} = (\beta R)^{-1} c_t^{-\gamma}$$

that implies a constant consumption path across time

$$c_{t+1} = c_t$$

Then, we can compute the implied savings function ???????? Discuss the computation of the savings function assuming a constant savings rule as a function of the individual state variables $c_t = g(A_t, y_t)$, or reduce it to a single state variable, that is $c_t = g(A_t)$.

4.6.3 The Random Walk Theory of Stock Prices

Here we interpret the asset as a share of an enterprise that sells for price p_t measured in terms of consumption goods at period t per share during period t and pays a non-negative random dividend of d_t consumption goods to the owner of the share at the beginning of t. We assume that d_t is governed by a time invariant Markov process. Let s_t denote the number of shares owned by the consumer at the start of t. The implied budget constraint is given by

$$c_t + p_t s_{t+1} = y_t + s_t(p_t + d_t)$$

or

$$c_t = y_t + (s_t - s_{t+1})p_t + s_t d_t,$$

where

- $(s_t - s_{t+1}) < 0$ the number of assets tomorrow is increasing therefore c_t is decreasing. The consumer is lending resources.
- $(s_t - s_{t+1}) > 0$ the number of assets tomorrow is decreasing therefore c_t is increasing. The consumer is borrowing resources.

The implied return is given by $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$.The Euler equation that solves the problem is

$$p_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}(p_{t+1} + d_{t+1})]$$

For any two random variables x, y we have the formula $E(xy) = E(x)E(y) + cov(x, y)$ where $cov(x, y) = E(x - E(x))(y - E(y))$ is the conditional covariance. Applying this formula in the above expression we have

$$p_t = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)}\right] E_t\left[p_{t+1} + d_{t+1}\right] + \beta cov_t \left(\frac{u'(c_{t+1})}{u'(c_t)}, (p_{t+1} + d_{t+1})\right)$$

To obtain the random walk theory of stock prices, it is necessary to make some assumptions:

1. $cov_t \left(\frac{u'(c_{t+1})}{u'(c_t)}, (p_{t+1} + d_{t+1})\right) = 0$
2. $E_t \left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = 1$, or constant. For these statement to be true it is sufficient that $u(c_t)$ is linear in c_t, so that $u'(c_t)$ is independent of c_t.

The resulting expression implies

$$p_t = \beta E_t[p_{t+1} + d_{t+1}]$$

or rearranging terms

$$E_t[p_{t+1} + d_{t+1}] = \beta^{-1} p_t$$

If the future price and dividends follow a first-order Markov process, no other variable in the information set can be used to predict the future returns. Using the law of iterated expectations

$$\begin{aligned} p_t &= \beta E_t[p_{t+1} + d_{t+1}] \\ p_{t+1} &= \beta E_{t+1}[p_{t+2} + d_{t+2}] \\ p_{t+2} &= \beta E_{t+2}[p_{t+3} + d_{t+3}] \end{aligned}$$

we obtain

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j d_{t+j}$$

or

$$p_{t+1} = \beta^{-1} E_t \sum_{j=2}^{\infty} \beta^j d_{t+j}$$

It is direct to show that this expression satisfies the first-order stochastic difference equation. We just need to substitute in the above expression. Formally,

$$\begin{aligned} E_t \sum_{j=1}^{\infty} \beta^j d_{t+j} &= \beta E_t[\beta^{-1} E_t \sum_{j=2}^{\infty} \beta^j d_{t+j} + d_{t+1}] \\ \beta E_t d_{t+1} + E_t \sum_{j=2}^{\infty} \beta^j d_{t+j} &= \beta E_t[\beta^{-1} E_t \sum_{j=2}^{\infty} \beta^j d_{t+j} + d_{t+1}] \end{aligned}$$

Clearly this is a solution to the equation. The main problem is that there exist a general class os solution that also satisfy this equation

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j d_{t+j} + \varepsilon_t \left(\frac{1}{\beta}\right)^t$$

where ε_t denotes a random variable that satisfies $E_t[\varepsilon_{t+1}] = \varepsilon_t$. This expression relates the price of a share p_t as the discounted expected dividends and a bubble term not related with the fundamental variables of the economy. We leave to the reader to show that the general equation also satisfied the first-order stochastic difference equation. In the next section we will show that using general equilibrium we will derive a transversality condition that will rule that out.

4.6.4 Recursive Formulation of the Lucas Asset Pricing Model

In more general versions of the Lucas' model, the asset pricing function is a key object that need to be solved for. In order to make the conditional expectation $u'(d_t)p_t = E_t[\beta u'(d_{t+1})(p_{t+1} + d_{t+1})]$ well defined, the representative agents needs to have a law of motion over time that maps d_t into p_t. Given that the expectation is calculated using the perceived pricing function, the notion of a rational expectation equilibrium is that the actual pricing function equals the perceived pricing function used to form expectations. In this section we study the nature of the mapping from perceived to actual pricing functions induced by the Euler equations. The sequential optimization problem is given by

$$\max_{\{c_t, s_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} s.to. \qquad c_t + p_t s_{t+1} &= s_t(p_t + d_t) \\ s_{t+1} &\geq -B \end{aligned}$$

In order to have a well posed problem, we must posit a law of motion for the stock price. This step is necessary so that the constraints of the problem are fully spelled out, and the conditional expectation is well defined. The pricing function is given by

$$p_t = h(x_t),$$

where h is a continuos, bounded function defined on the domain of the current state x_t. In this particular formulation, the aggregate state is given by the dividend process. The transition law $F(x', x)$ for dividends, together with the pricing function define the perceived law of motion for the price of trees. The recursive formulation implies

$$v(s_t; d_t) = \max_{\{s_{t+1}\}} \{u(s_t(p_t + d_t) - p_t s_{t+1}) + \beta E_t v(s_{t+1}; d_{t+1})\}$$

where $p_t = h(d_t)$. If we use a prime to denote future values, and substitute the perceived pricing function in the consumer problem, we can rewrite the problem as

$$v(s[h(d)+d]) = \max_{\{s'\}}\{u(s[h(d)+d]-h(d)s')+\beta \int v(s'[h(d')+d'])dF(d', d)\}.$$

The first-order condition associated with the problem is

$$-u'(s[h(d)+d]-h(d)s')h(d)+\beta \int [h(d')+d']v'(s'[h(d')+d'])dF(d', d) = 0,$$

Using the envelope condition, we can compute the change of the value function associated to a change on the share holdings s. We know that

$$v'(s[h(d) + d]) = u'(c)[h(d) + d],$$

where $c(d) = s[h(d) + d] - s'h(d)$. If we updated one period we have $v'(s[h(d') + d']) = u'(c')[h(d') + d']$, and we can rewrite the Euler equations as

$$h(d)u'[c(d)] = \beta \int [h(d') + d']u'[c(d')]dF(d', d)$$

or

$$h(d)u'[c(d)] = \beta \int h(d')u'[c(d')]dF(d', d)+\beta \int d'u'(c(d'))dF(d', d).$$

We can define a function $w(d)$ as

$$w(d) = h(d)u'[c(d)]$$

that depends on the evolution of the dividend process,

$$w(d) = \beta \int w(d')dF(d',d) + \beta \int d'u'(c(d'))dF(d',d)$$

Now we can impose the equilibrium conditions on the Lucas tree model.

1. There is no trade, and one tree per person $s = s' = 1$,
2. Households only consume the fruit of the tree, that is $c(d) = s[h(d) + d] - s'h(d) = d$

If we substitute the equilibrium condition on the Euler equation we have

$$w(d) = \beta \int w(d')dF(d',d) + \beta \int d'u'(d')dF(d',d)$$

This is a functional equation in the unknown function $w(d) = h(d)u'(d)$. Because $u(d)$ is known, once $w(d)$ has been determined, we can compute the implied pricing function $h(d) = w(d)/u'(d)$ that is the goal of the model. The objective is to solve the functional equation for $w(d)$ that is approached by iterating on $w^j(d)$. We ignore all the implies math that show that the functional equation is a contraction and has a unique fixed point. We want to focus on the computation aspects to obtain solutions.

1. We define a function that does not depend on $w(d)$,

$$g(d) \equiv \beta \int d'u'(d')dF(d',d),$$

2. Then, the functional equation becomes

$$w(d) = g(d) + \beta \int w(d')dF(d',d)$$

3. We start iterating on $w^j(d)$ as follows

$$w^{j+1}(d) = g(d) + \beta \int w^j(d')dF(d',d)$$

4. Starting from any initial continuous and bounded function $w^0(d)$, we can compute $w^1(d)$, if we don't have the same function we update it with the new one, and we iterate until the functional equation satisfies the convergence criterion sup $norm\ |w^{j+1}(d) - w^j(d)| < \varepsilon$. Once we know the limiting function $w(d)$ is known, the pricing function can be easily calculated

$$h(d) = \frac{w(d)}{u'(d)},$$

We could use an alternative approach where we iterate on the pricing function. We sketch the solution method for the alternative approach.

1. Guess a pricing function $h^0(d)$, and solve the pricing equation

2. Then, we obtain a new functional equation

$$h^1(d)u'(d) = g(d) + \beta \int h^0(d')u'(d')dF(d', d),$$

and more generally,

$$h^{j+1}(d)u'(d) = g(d) + \beta \int h^j(d')u'(d')dF(d', d)$$

This equation can be regarded as a mapping a perceived pricing function $h^j(d)$ into an actual pricing function $h^{j+1}(d)$. A rational expectations equilibrium is a fixed point of this mapping from perceived pricing functions to actual pricing functions

Example: Dividend growth

Next, we assume that dividends grow according to a stochastic process, $d_{t+1} = \lambda_{t+1}d_t$, where λ_t follows a Markov process with a transition matrix $F(d'/d)$. If the utility function has constant relative risk aversion, then the pricing equation satisfies

$$p_t = E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(p_{t+1} + d_{t+1})\right].$$

Dividing by the dividend in both sides d_t,

$$\frac{p_t}{d_t} = E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(\frac{p_{t+1}+d_{t+1}}{d_t})\right]$$

knowing that the equilibrium condition implies $\lambda_{t+1} = \frac{c_{t+1}}{c_t}$, and then $d_t = \frac{d_{t+1}}{\lambda_{t+1}}$,

$$\frac{p_t}{d_t} = E_t\left[\beta\left(\lambda_{t+1}\right)^{1-\gamma}(1+\frac{p_{t+1}}{d_{t+1}})\right]$$

we can arrange the expression defining the price-dividend ratio, $w_i = \frac{p_t}{d_t}$. If the growth rate can only take finite values j, then we have

$$w_i = \beta\sum_j \pi_{ij}\lambda_j^{1-\gamma}(1+w_j)$$

where the stochastic discount factor becomes $m_j = \lambda_j^{1-\gamma}$. If we consider a two state Markov chain

$$\pi_{ij} = \begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{21} & \pi_{22}\end{bmatrix} = \begin{bmatrix}0.25 & 0.75\\ 0.75 & 0.25\end{bmatrix}$$

assume two different values for consumption growth $\lambda_1 = 1.04$ and $\lambda_2 = 1.02$.

4.6.5 Contingent Claim Market with Continuous State Space

We show how to use this model to price claims to virtually all imaginable assets. We begin by pricing one-period state contingent securities, and then, we will show how to derive j-step-ahead state contingent securities. We consider the Lucas tree model together with state contingent commodities. Let x be the aggregate state of the economy. We assume that the state evolves according to a Markov process described by $f(x', x)$. The representative consumer solves

$$\max E_0\sum_{t=0}^{\infty}\beta^t u(c_t)$$

$$s.to. \qquad c_t + p_t s_{t+1} + \int q(x_{t+1}, x_t) y(x_{t+1}) dx_{t+1} = s_t(p_t + d_t) + y(x_t),$$

where $y(x_{t+1})$ denotes the net amount of the date $t+1$ consumption good, contingent on having the aggregate state x_{t+1}. The consumer pays a price $q(x_{t+1}, x_t)$ given that today's state in the economy is x_t. As usual, we can express the price of a share as a function of the current state of the economy $p_t = p(x_t)$ and $d_t = d(x_t)$. The Bellman equation is given by

$$v(\underbrace{[p+d]s + y(x)}_{\text{Wealth}}, \underbrace{x}_{\text{Aggregate State}}) = \max_{c,s',y(x')} \{u(c) + \beta E v([p'+d']s' + y(x'), x')\}$$

$$s.to. \qquad c + ps' + \int q(x', x') y(x') dx' = [p+d]s + y(x),$$

The first-order conditions of the consumer problem with respect to the control variables $\{c, s', y(x')\}$ are given by

$$u'(c) = \lambda$$

$$\beta \int v_1([p'+d']s' + y(x'), x')(p'+d') f(x', x) dx' = \lambda$$

$$\beta v_1([p'+d']s' + y(x'), x')) f(x', x) = \lambda q(x', x')$$

using the envelope condition $v_1(s', x) = u(c)$. Combining this expressions we have

$$q(x', x) = \beta \frac{u'(c(x'))}{u'(c(x))} f(x', x)$$

using market equilibrium $c = d = x$, where x is the level of dividends and the state of the economy. So we have,

$$q(x', x) = \beta \frac{u'(x')}{u'(x)} f(x', x)$$

Now, we want to show how to price assets using the pricing kernel. We study the following examples.

- Consider a function $w(x')$ that assigns values at state x'. The price of a claim that pays off the state-contingent amount $w(x')$ in next period state is given by

$$\int \underbrace{w(x')}_{\text{State contingent payoff}} \underbrace{q(x', x)}_{\text{Pricing Kernel}} dx' = \int w(x') \beta \frac{u'(x')}{u'(x)} f(x', x) dx'$$

- Consider a risk-less asset that always has the same payoff $w(x') = 1$. We use the pricing kernel to price the asset

$$\int q(x', x)dx' = \beta \int \frac{u'(x')}{u'(x)} f(x', x)dx' = \frac{1}{R}$$

- Consider a risky asset with the following payoff structure

$$w(x') = \begin{cases} 1 & \text{if } x' \leq \overline{x} \\ 0 & \text{if } x' > \overline{x} \end{cases}$$

the asset price is given by

$$\int_{x' \leq \overline{x}} q(x', x)dx' = \beta \int_{x' \leq \overline{x}} \frac{u'(x')}{u'(x)} f(x', x)dx'$$

4.7 Term Structure of Interest Rates

The term structure of interest rates or yield curve is commonly defined as a collection of yields to maturity for bonds with different rates of maturity. Next, we modify the Lucas model to study the determination of the term structure of interest rate. We suppose that there are markets in one- and two period perfectly safe loans, which bear gross rates of returns R_{1t} and R_{2t} are know with certainty and risk free from the view point of the agents. Both prices are denominated in units of time t consumption goods. The representative agent solves

$$\max_{\{c_t, s_{t+1}, L_{1t+1}, L_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.to. \qquad c_t + L_{1t+1} + L_{2t+1} + p_t s_{t+1} = s_t(p_t + d_t) + R_{1t} L_{1t1} + R_{2t} L_{2t-1},$$

Evidently, the solution of the consumer problem is autarky $c_t = d_t$, but we cannot impose equilibrium before taking the consumer first-order conditions. In as sense, this is a model where agents do not trade in equilibrium. However, we want to use it to price assets with different term structures. For this reason, we can simplify the asset structure and only consider the riskless assets assuming no trade on shares (which we know will happen in equilibrium)

$$\max_{\{c_t, L_{1t+1}, L_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.to. \qquad c_t + L_{1t+1} + L_{2t+1} = d_t + R_{1t} L_{1t1} + R_{2t} L_{2t-1},$$

The stochastic Lagrangian of the consumer problem is given by

$$J = E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t (d_t + R_{1t} L_{1t1} + R_{2t} L_{2t-1} - c_t - L_{1t+1} - L_{2t+1})]$$

where $\{\lambda_t\}$ is a sequence of random Lagrange multipliers. The first-order conditions with respect to $\{c_t, L_{1t}, L_{2t}\}$ are given by

$$u'(c_t) - \lambda_t = 0$$
$$-\lambda_t + \beta E_t \lambda_{t+1} R_{1t} = 0$$
$$-\lambda_t + \beta^2 E_t \lambda_{t+2} R_{2t} = 0$$

Combining the first-order conditions implies

$$E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{1t} \right] = 1$$
$$E_t \left[\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} R_{2t} \right] = 1$$

Given that both assets are riskless, we can rewrite the Euler equations. Formally,

$$R_{1t}^{-1} = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)}$$
$$R_{2t}^{-1} = E_t \beta^2 \frac{u'(c_{t+2})}{u'(c_t)}$$

To derive the term structure we have to manipulate these expressions. Another way to represent R_{2t}^{-1} is

$$R_{2t}^{-1} = E_t \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} \frac{u'(c_{t+1})}{u'(c_{t+1})} = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}.$$

Using the law of iterated expectations we obtain

$$R_{2t}^{-1} = E_t E_{t+1} \beta \frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \underbrace{E_{t+1} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}}_{R_{1t+1}^{-1}}.$$

that is

$$R_{2t}^{-1} = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{1t+1}^{-1}.$$

Finally, we can use the definition of conditional covariance to obtain

$$R_{2t}^{-1} = \underbrace{E_t \beta \frac{u'(c_{t+1})}{u'(c_t)}}_{R_{1t}^{-1}} E_t R_{1t+1}^{-1} + cov_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, R_{1t+1}^{-1} \right],$$

so we obtain

$$R_{2t}^{-1} = R_{1t}^{-1} E_t R_{1t+1}^{-1} + cov_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, R_{1t+1}^{-1} \right]$$

This expression is a generalized version of the expectations theory of the term structure of interest rates, adjusted for the risk premium. The classical theory of the term structure of interest rates is that the long term interest rates should be determined by expected future short-term interest rates. For example, the pure expectations theory states that

$$R_{2t}^{-1} = R_{1t}^{-1} E_t R_{1t+1}^{-1}$$

According to our modified formula, the pure expectations theory only holds in special cases.

1. The utility function is linear in consumption, $u(c) = Ac$, so marginal utility is constant $u'(c_{t+1})/u'(c_t) = 1$. Therefore, $R = \beta^{-1}$ and $cov_t(\cdot) = 0$. Consequently, the yield curve is given by

$$R_{2t}^{-1} = R_{1t}^{-1} R_{1t+1}^{-1} = \beta^2,$$

2. Another case is in the absence of uncertainty ($cov_t(\cdot) = 0$) in the dividend. Therefore, we have the same result $R_{2t} = \beta^{-2}$.

   The general expression for the price of a time t bond that yields one unit of the consumption good at period $t+i$ is

$$R_{it}^{-1} = E_t \beta^i \frac{u'(c_{t+i})}{u'(c_t)}$$

   Next, we explore some special cases.

   Example 1: Zero Coupon Bond

   Consider a zero coupon bond that only pays at maturity with a coupon payment of one unit of consumption $p_t = 1/R_t^j$. We use the fact that in equilibrium $c = d$. The yield to maturity is given by

$$R_{jt} = \left[\frac{1}{p_t}\right]^{\frac{1}{j}} = \left[\frac{1}{\beta^j E_t[\frac{u'(d_{t+j})}{u'(d_t)}]}\right]^{\frac{1}{j}} = \beta^{-1}\left[u'(d_t)E_t[u'(d_{t+j})]^{-1}\right]^{\frac{1}{j}}$$

   Example 2: I.I.D. Case

   Consider iid shocks. The yield to maturity between two different bonds with expiration at time k and j are given by

$$\frac{R_{jt}}{R_{kt}} = \frac{[u'(d_t)E_t[u'(d_{t+j})]^{-1}]^{\frac{1}{j}}}{[u'(d_t)E_t[u'(d_{t+k})]^{-1}]^{\frac{1}{k}}} = \left[u'(d_t)E_t[u'(d_{t+j})]^{-1}\right]^{\frac{1}{j}}\left[u'(d_t)E_t[u'(d_{t+k})]^{-1}\right]^{-\frac{1}{k}}$$

   because shocks are iid, we know that $E_t[u'(d_{t+j})] = E_t[u'(d_{t+k})] = E(d)^{-1}$. Therefore, we have

$$R_{jt} = R_{kt}\left[u'(d_t)E[u'(d)]^{-1}\right]^{\frac{k-j}{kj}}$$

   In this case the term structure is upward slopping when $u'(d_t)$ is less than $Eu'(d)$, that is when consumption is relatively high today with low marginal utility, and agents would like to save for the future. In equilibrium, the short-term interest rate (?).

   Example 3: Persistent dividend shock and logarithmic utility

Consider a utility function of the form $u(c) = \ln(c)$, and dividends follow the stochastic process

$$d_{t+1} = \rho d_t \theta_{t+1}$$

where $\rho > 0$ and θ_{t+1} is a sequence of independently and identically distributed random variables that are positive with probability 1. Now, we can complete the model imposing equilibrium conditions $c_t = d_t$ for all t. Formally,

$$R_{1t}^{-1} = E_t \beta \frac{u'(d_{t+1})}{u'(d_t)}$$

$$R_{2t}^{-1} = E_t \beta^2 \frac{u'(d_{t+2})}{u'(d_t)}$$

The imply equations are given by

$$R_{1t}^{-1} = E_t \beta \frac{d_t}{d_{t+1}}$$

$$R_{2t}^{-1} = E_t \beta^2 \frac{d_t}{d_{t+2}}$$

replacing the dividend process

$$R_{1t}^{-1} = E(\frac{\beta}{\rho})(\theta^{-1})$$

$$R_{2t}^{-1} = \left[E(\frac{\beta}{\rho})(\theta^{-1}) \right]^2$$

where we are using the independence over time of θ_t. The level of interest rate raises with the term of maturity if $\rho/[\beta E(\theta^{-1})] > 1$ and falls is $\rho/[\beta E(\theta^{-1})] < 1$.

4.8 Pricing Functions in the Presence of Multiple Stocks

Next, we are interested on deriving the pricing function in the presence of different type of trees that yield different quantities of fruits. There n kinds of trees, and each household is initially endowed with one of each kind of tree. The aggregate dividend is given $d_t = \sum_{i=1}^{n} d_{it}$. The representative consumer maximizes

$$\max_{c_t, \{s_{it+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.to. \qquad c_t + \textstyle\sum_{i=1}^{n} p_{it} s_{it+1} = \sum_{i=1}^{n} s_{it}(p_{it} + d_{it})$$
$$s_{it+1} \geq -B$$

The Euler equation for the ith stock is given by

$$p_{it} = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (p_{it+1} + d_{it+1}) \right],$$

in equilibrium the optimal plan implies $c_t = d_t$. The pricing equation becomes

$$p_{it} = \beta E_t \left[\frac{u'(d_{t+1})}{u'(d_t)} (p_{it+1} + d_{it+1}) \right],$$

Next we consider a specialized example that allows to characterize the pricing equation

Example: Logarithmic utility

If the utility function is $u(c) = \ln c$, we have

$$p_{it} = \beta E_t \left[\frac{d_t}{d_{t+1}} (p_{it+1} + d_{it+1}) \right],$$

Next, we use a guess and verify method to determine the pricing function. We assume a time-varing linear pricing function of the form

$$p_{it} = \phi_{it} d_t,$$

we need to determine the coefficients of the pricing function, ϕ_{it} for every asset. We can compute the coefficient using the method of undetermined coefficients.

$$\phi_{it} d_t = \beta E_t \left[\frac{d_t}{d_{t+1}} (\phi_{it+1} d_{t+1} + d_{it+1}) \right],$$

$$\phi_{it} = \beta E_t \left[\phi_{it+1} + \frac{d_{it+1}}{d_{t+1}} \right]$$

or

$$\phi_{it} = \beta E_t \phi_{it+1} + \beta E_t \left[\frac{d_{it+1}}{d_{t+1}} \right]$$

iterating we obtain

$$\phi_{it} = E_t \sum_{j=1}^{\infty} \beta^i \frac{d_{it+j}}{d_{t+j}},$$

that determines the pricing function in terms of the conditional distribution of the stochastic process $\{d_{it}, d_t\}$. We consider two examples.

- $n = 1$: We only have one asset, that is $d_{1t} = d_t$ for all t. We obtain the standard pricing as a special case.

$$\phi_{1t} = E_t \sum_{j=1}^{\infty} \beta^i \frac{d_{1t+j}}{d_{t+j}} = \frac{\beta}{1-\beta},$$

- $n = 2$: Now, we have two assets. We consider a special case for the individual process. We assume that both assets are negatively related. In particular,

$$d_{1t} = \frac{(1-\epsilon_t)}{2} d_t,$$
$$d_{2t} = \frac{(1+\epsilon_t)}{2} d_t,$$

where ϵ_t is a random variable distributed between 0 and 1, and follows a Markov process $E_t \epsilon_{t+j} = \rho^j \epsilon_t$ where $|\rho| < 1$. That implies

$$E_t d_{1t+j} = \frac{(1-\rho^j \epsilon_t)}{2} d_t,$$
$$E_t d_{2t+j} = \frac{(1+\rho^j \epsilon_t)}{2} d_t,$$

Using this fact we obtain for asset 1,

$$\phi_{1t} = E_t \sum_{j=1}^{\infty} \beta^i \left[\frac{d_{1t+j}}{d_{t+j}} \right] = E_t \sum_{j=1}^{\infty} \beta^i \left[\frac{(1-\epsilon_t)}{2} \right] = \sum_{j=1}^{\infty} \beta^i \frac{(1-\rho^j \epsilon_t)}{2}$$
$$\phi_{1t} = \frac{1}{2} \left[\frac{\beta}{1-\beta} - \frac{\rho\beta}{1-\rho\beta} \epsilon_t \right]$$

where

$$\phi_{2t} = \frac{1}{2}\left[\frac{\beta}{1-\beta} + \frac{\rho\beta}{1-\rho\beta}\epsilon_t\right]$$

These are the coefficients of the time-varing pricing functions. An alternative approach would been to assume two-state variables $x = (\epsilon, d)$, and we would have a time-invariant pricing function. We assume a quadratic pricing function of the form

$$p_i = (a_i - b_i\epsilon)d$$

where we have that $E\epsilon' = \rho\epsilon$, and the pricing function is given by

$$p_i = \beta E\frac{u'(d')}{u'(d)}(p_i' + d_i'),$$

We use the method of undetermined coefficients to compute the price function. If we substitute the guess function into the Euler equation we obtain

$$\begin{aligned}(a_i - b_i\epsilon)d &= \beta E\left[\frac{d}{d'}((a_i - b_i\epsilon)d' + d_i')\right],\\ a_i - b_i\epsilon &= \beta E\left[(a_i - b_i\epsilon) + \frac{d_i'}{d'}\right]\end{aligned}$$

and we know that $\frac{d_1'}{d'} = \frac{1-\epsilon}{2}$ and $\frac{d_2'}{d'} = \frac{1+\epsilon}{2}$, so we have

$$\begin{aligned}a_1 - b_1\epsilon &= \beta E\left[(a_1 - b_1\epsilon) + \frac{1-\epsilon}{2}\right],\\ a_1 - b_1\epsilon &= \beta(1/2 + a_1) - \beta(b_1 + 1/2)E(\epsilon)\\ a_1 - b_1\epsilon &= \beta(1/2 + a_1) - \beta(b_1 + 1/2)\rho\epsilon\end{aligned}$$

so we obtain $a_1 = \beta/2(1-\beta)$ and $b_1 = \beta\rho/(1-\beta\rho)$.

4.9 Modigliani-Miller Theorem

The Modigliani-Miller Theorem shows that under certain circumstances the total value of a firm is independent of the firm's financial structure. We study the M-M theorem in a Lucas tree model economy. Latter on, we will study it in an economy with production.

Consider an agent whose only asset is a tree that yields an stochastic crop y at each period. Suppose the agent want to sell the firm to the public, and faces two different alternatives.

1. Issue a number of bonds $B > 0$ that pay a riskless return r or coupon at every period. To avoid bankruptcy issues, the agent has to ensure that the return can be repaid in all states of nature. Formally, $rB < \underline{y}$.
2. An alternative implies issuing shares entitled to the residual crop, where the dividend is the residual. Formally,

$$d_t = \frac{(y_t - rB)}{S},$$

The equilibrium prices can be easily be computed using the pricing kernel used to price contingent claims. Notice that this are not one period claims. They yield payments and infinite number of periods, therefore they need to be computed correctly,

$$\begin{aligned} p_t^B &= \sum_{i=1}^{\infty} \int rq(x_{t+i}, x_t) dx_{t+i} \\ p_t^S &= \sum_{i=1}^{\infty} \int \frac{(y_{t+i} - rB)}{S} q(x_{t+i}, x_t) dx_{t+i} \end{aligned}$$

The total value of the firm is given by

$$\begin{aligned} p_t^B B + p_t^S S &= \sum_{i=1}^{\infty} \int rq(x_{t+i}, x_t) dx_{t+i} + \sum_{i=1}^{\infty} \int \frac{(y_{t+i} - rB)}{S} q(x_{t+i}, x_t) dx_{t+i} \\ &= \sum_{i=1}^{\infty} \int y_{t+j} q(x_{t+i}, x_t) dx_{t+i}, \end{aligned}$$

using the formula for the pricing kernel we obtain

$$p_t = E_t \sum_{i=1}^{\infty} \beta^j \frac{u'(y_{t+i})}{u'(y_t)} y_{t+i}$$

The value of the firm is independent of the financing scheme, because the equilibrium prices of bonds and shares will adjust to reflect the inherent riskiness of the financial structure. Clearly, the value of the firm is independent of the number of bonds B or the coupon rate r.

Example 1:

Assume that preferences are $u(c) = \ln c_t$ and y_{t+i} is $i.i.d.$ shock over time so that $E(y_{t+i}) = E(y)$, and $1/y_{t+i}$ is also iid. With this preferences we know that the price of a tree is given by

$$p_t = \frac{\beta}{1-\beta} y_t,$$

and the bond price is

$$p_t^B = E_t \sum_{i=1}^{\infty} \left[r\beta^j \frac{u'(y_{t+i})}{u'(y_t)} \right] = \frac{r\beta}{1-\beta} E(y^{-1}) y_t$$

or using the price of a tree $y_t = (1-\beta)p_t/\beta$, we can write is as

$$p_t^B = rE(y^{-1})p_t$$

where as the price of a share is

$$p_t^S = E_t \sum_{i=1}^{\infty} \left[\frac{(y_{t+i} - rB)}{S} \beta^j \frac{u'(y_{t+i})}{u'(y_t)} \right] = E_t \sum_{i=1}^{\infty} \left[(1 - \frac{rB}{y_{t+i}}) \beta^j \frac{y_t}{S} \right]$$
$$= \frac{\beta}{1-\beta} [1 - rBE(y^{-1})] \frac{y_t}{S}$$

or

$$p_t^S = [1 - rBE(y^{-1})] \frac{p_t}{S}$$

The value of the firm satisfies the Modigliani-Miller theorem

$$p_t^B B + p_t^S S = rBE(y^{-1})p_t + [1 - rBE(y^{-1})] \frac{p_t}{S} S = p_t$$

the value of the firm is independent of the financial structure. The price of an issued share depends negatively on the number of bonds B, the coupon r, and the number of issued shares. The price of a bond depends on the coupon yield. Now we want to relate the expected return of the assets to the level of riskiness.

First, it is direct to show that the capital gains on either bonds or stocks is related to the expected growth of the tree $E_t(g_{t+1}) = E_t(y_{t+1}/y_t)$. Clearly all have the same expected growth,

$$E_t(p_{t+1}/p_t) = E_t\left[\frac{\frac{\beta}{1-\beta}y_{t+1}}{\frac{\beta}{1-\beta}y_t}\right]$$

$$E_t(p^B_{t+1}/p^B_t) = E_t\left[\frac{\frac{r\beta}{1-\beta}E(y^{-1})y_{t+1}}{\frac{r\beta}{1-\beta}E(y^{-1})y_t}\right]$$

$$E_t(p^S_{t+1}/p^S_t) = E_t\left[\frac{\frac{\beta}{1-\beta}[1 - rBE(y^{-1})]\frac{y_{t+1}}{S}}{\frac{\beta}{1-\beta}[1 - rBE(y^{-1})]\frac{y_t}{S}}\right]$$

Second, any difference in expected rates of return must arise from the expected yields due to next period dividends and coupons. Bonds are riskless, whereas shares are riskly assets. Formally, the expected return of a bond is

$$E_t\left[\frac{r}{p^B_t}\right] = \frac{r}{p^B_t} = \{1 - E_t(y_{t+1})E_t(y^{-1}_{t+1}) + E_t(y_{t+1})E_t(y^{-1}_{t+1})\}\frac{r}{p^B_t}$$

$$= [1 - E(y)E(y^{-1}) + E(y)E_t(y^{-1})]\frac{r}{\frac{r\beta}{1-\beta}E(y^{-1})\frac{(1-\beta)p_t}{\beta}}$$

$$\frac{r}{p^B_t} = \frac{1 - E(y)E(y^{-1})}{p_tE(y^{-1})} + \frac{E_t(y_{t+1})}{p_t} = \frac{1}{p_tE(y^{-1}_{t+1})} < E_t\left[\frac{y_{t+1}}{p_t}\right]$$

because of the Jensen inequality that states $E(y^{-1})$