Chapter 5

Dynamic Firms: Theory of Investment

5.1 Neoclassical Theory of Investment

We first start describing the neoclassical model of investment described by Jorgenson (1963). In the model firms produce output using two different inputs, capital k_t and labor l_t. The price of output is given by p_t. Labor services are hired in a spot labor market, and workers earn a wage w_t, whereas the size of the plant or firms is endogenously determined through retains earning. We assume that all the investment is internal, and we rule out any external financing at this point. We also assume that investment is not irreversible, therefore at any point in time firms can decrease their size or capital stock by disinvesting. The price of investment is p_t^I, and we assume that capital and output are the same good, so the slope of transformation is one, that is $p_t^I = p_t$.

Capital depreciates at a constant rate δ. Therefore, part of the investment will be used to repair the equipment that has been damage through the production process. The production function $Q = f(k, l)$ is neoclassical and satisfies

1. Constant returns to scale:

$$f(\lambda k, \lambda l) = \lambda f(k, l)$$

2. Essentiality of inputs:

$$f(0,l) = f(k,0) = 0$$

3. Diminishing marginal returns:

$$f_k > 0, \quad f_l > 0 \quad, f_{kk} < 0, \quad , f_{ll} < 0$$

4. Inada conditions:

$$\lim_{k \to \infty} f_k = 0$$
$$\lim_{k \to 0} f_k = \infty$$

We assume that firms have limited liability,

$$d_t = \max(f(k_t, l_t) - w_t l_t - x_t, 0),$$

where investment is $x_t = (1-\delta)k_t - k_{t+1}$. Firms maximize discounted dividends. Formally,

$$v(k_0) = \max_{\{l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t d_t,$$

or

$$v(k_0) = \max_{\{l_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} R^{-t}(p_t f(k_t, l_t) - w_t l_t - p_t^I x_t),$$

$$s.t. \qquad k_{t+1} - (1-\delta)k_t = x_t,$$

We impose a terminal condition that forces the present value of assets in the firm to be non negative. That is $\lim_{t\to\infty} R^{-t} K_{t+1} = 0$. Let q_t be the Lagrange multiplier of the investment constraint. This multiplier measures the contribution of a unit of capital at time t to the value of the firm as measured at time 0. This is the present value shadow price because it measures the value with respect time zero. To measure investment in terms of the present value of the firm we multiply the expression by R^{-t}. The first-order conditions with respect to l_t, x_t, and k_{t+1} are given by

$$R^{-t}[p_t f_l(k_t, l_t) - w_t] = 0$$
$$-R^{-t} p_t^I + q_t R^{-t} = 0$$
$$-R^{-t} q_t + R^{-(t+1)}(p_{t+1} f_k(k_{t+1}, l_{t+1}) + q_{t+1}(1-\delta)) = 0$$

together with a transversality condition states that $\lim_{T\to\infty} R^{-T} q_T k_{T+1} = 0$. The first condition states that the firms will hire labor until its marginal product (or benefit) equals the wage rate. Hence, the labor demand function is negatively related to the wage rate. The second condition states that the current (shadow) value of a unit of investment is worth exactly its cost, p_t^I. The implied investment rule determines that if the marginal revenue from an extra unit of capital that cost one is larger than one, then firms should invest positive amounts. Firms will invest or disinvest until the second condition holds with equality. Finally, the last condition states that firms will invest/or disinvest until the internal rate of return from an extra unit of capital equals the return of an alternative asset. Rearranging terms we have

$$f_l(k_t, l_t) = \frac{w_t}{p_t},$$
$$p_t^I = q_t,$$

and

$$Rq_t = p_{t+1} f_k(k_{t+1}, l_{t+1}) + q_{t+1}(1-\delta),$$

together with

$$k_{t+1} - x_t = (1-\delta) k_t,$$

If we normalize the price of output $p_t = 1$ in all periods, and the technology of transformation is $p_t = p_t^I$. Then, we can rewrite the first-order conditions as

$$q_t = 1 \qquad \forall t,$$

and

$$R = f_k(k_t, l_t) + 1 - \delta,$$

This last equation is the standard condition for static firms, and the price of capital investment is on, this is the marginal q. Investment is a function of the existing capital stock and the real interest rate. Formally,

$$x_t = k_{t+1} - (1-\delta)k_t$$

the optimal savings policy $R = 1 - \delta + f_k(k_{t+1}, l_{t+1})$, that determines an implicit function for tomorrow decision rules where $l_{t+1} = \varphi(w_t, k_{t+1})$. That is

$$R = 1 - \delta + f_k(k_{t+1}, \varphi(w_t, k_{t+1}))$$

so the optimal policy depends on the relative prices

$$k_{t+1} = g(R, w)$$

Substituting into the investment function

$$x_t = g(R, w) - (1-\delta)k_t$$

The value of a firm at a given point in time is captured by q_t that determines the value of the current capital stock. Formally, we multiply the last expression by k_{t+1}

$$q_t k_{t+1} R = f_k(k_{t+1}, l_{t+1})k_{t+1} + q_{t+1}(1-\delta)k_{t+1}$$

The Euler theorem implies $f(k, l) = kf_k + lf_l$, so we have

$$q_t k_{t+1} R = f(k_{t+1}, l_{t+1}) - w_{t+1}l_{t+1} + q_{t+1}(1-\delta)k_{t+1}$$

we can rewrite the investment equation as follows $q_{t+1}(k_{t+2} - x_{t+1}) = q_{t+1}(1-\delta)k_{t+1}$. If we substitute the expression in the above equation we obtain

$$q_t k_{t+1} = R^{-1}[f(k_{t+1}, l_{t+1}) - w_{t+1}l_{t+1} - q_{t+1}x_{t+1} + q_{t+1}k_{t+2}]$$

where $d_{t+1} = f(k_{t+1}, l_{t+1}) - w_{t+1}l_{t+1} - q_{t+1}x_{t+1}$ substituting equations recursively

$$\begin{aligned} q_t k_{t+1} &= R^{-1}d_{t+1} + R^{-1}q_{t+1}k_{t+2} \\ q_{t+1}k_{t+2} &= R^{-1}d_{t+2} + R^{-1}q_{t+2}k_{t+3} \\ q_{t+2}k_{t+2} &= R^{-1}d_{t+3} + R^{-1}q_{t+3}k_{t+4} \end{aligned}$$

we have

$$q_t k_{t+1} = R^{-1}d_{t+1}+R^{-2}d_{t+2}+R^{-3}d_{t+3}+...+R^{-T}d_{t+T}+R^{-T}q_{t+T}k_{t+T+1},$$

adding up

$$q_t k_{t+1} = \sum_{t=0}^{T} R^{-t}[f(k_{t+1}, l_{t+1})-w_{t+1}l_{t+1}-q_{t+1}x_{t+1}]+R^{-T}q_{t+T}k_{t+T+1},$$

taking the limit

$$q_t k_{t+1} = \sum_{t=0}^{\infty} R^{-t}[f(k_{t+1}, l_{t+1})-w_{t+1}l_{t+1}-x_{t+1}]+\lim_{T\to\infty} R^{-T}q_{t+T}k_{t+T+1},$$

where the transversality condition takes care of the last term. Then,

$$q_t k_{t+1} = \sum_{t=0}^{\infty} R^{-t}d_t = v(k_t) = v_t$$

The value of the firm is given by

$$v_t = q_t k_{t+1},$$

but the Lagrange multiplier is constant, and normalized to $q_t = 1$, so the value of the firm is equal to the next period capital stock. Firms are priced based on their fundamental variables.

$$v_t = k_{t+1},$$

The marginal-q is always constant, and does not affect future investment decisions. That has posed some problems with the neoclassical theory of investment:

1. With discrete changes in r and w, firms will do all their investment after the change in prices. When prices do not change investment is zero once the optimal size has been attained.

2. In the presence of heterogeneous firms all the investment would be done in the firm with the highest marginal productivity.

3. Current investment is independent of future marginal products. Future change on business conditions should have an impact today. This does not happen with the neoclassical investment theory. We want a theory where firms are willing to smooth investment over time.

4. The investment is exogenously give to the firm. If we want to study its determination we need to include the consumer and solve for the market equilibrium.

5.2 Q-Theory of Investment: Internal adjustment costs

5.2.1 Marginal-q

To solve some of the problems with neoclassical theory of investment, several authors have modified the problem to have a theory that is more consistent with the data. They have done so by introducing internal adjustment cost. Now firms will have pay to install or uninstall capital in the firm.

Let $\varphi(\cdot)$ be the cost of installing one unit of capital in the firm. We assume that the function is increasing in investment x_t, that is the more we want to invest the more resources we have to use and therefore it is more expensive. Second, the function is decreasing in the capital stock. Firms that have a lot of physical capital because they have invested a lot in the past have lower installations costs. Formally, we have $\varphi(x_t/k_t)$ where $\varphi(0) = 0$, $\varphi'(\cdot) > 0$ and $\varphi''(\cdot) > 0$. The total cost of investing at time t is given by $x_t\varphi(x_t/k_t)$.

The firm optimization problem is given by

$$v(k_0) = \max_{\{l_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} R^{-t}(f(k_t, l_t) - w_t l_t - x_t - x_t\varphi(x_t/k_t)),$$

$$s.t. \qquad k_{t+1} - (1-\delta)k_t = x_t,$$

Let $R^{-t}q_t$ be the Lagrange multiplier of the investment constraint in terms of time t value. The first-order conditions with respect to l_t, x_t, and k_{t+1} are given by

$$R^{-t}(f_l(k_t, l_t) - w_t) = 0$$
$$-R^{-t}(1 + x_t\varphi(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi'(x_t/k_t)) + q_t R^{-t} = 0$$
$$-R^{-t}q_t + R^{-(t+1)}(f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}(1-\delta)) = 0$$

Rearranging terms

$$\begin{aligned} w_t &= f_l(k_t, l_t) \\ q_t &= 1 + x_t\varphi(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi'(x_t/k_t) \\ q_t R &= f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}(1-\delta) \end{aligned}$$

Next, we want to understand the nature of the investment decision in the problem. We explore how investment changes when marginal-q (q_t) changes. Formally,

$$q_t = 1 + x_t\varphi(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi'(x_t/k_t)$$

or we can rewrite it as $q_t = g(x_t/k_t)$. In this particular case, when $x_t = 0$, we have $q_t = g(0) = 1$. That is the neoclassical investment theory, which means when there is no investment there are no adjustment costs. Since the function is monotone, we can inverted and derive the investment decision as a function of marginal-q. Formally,

$$\frac{x_t}{k_t} = h(q_t),$$

and we already know that when $h(1) = 0$, and next we show that $h' > 0$

$$q_{(x/k)} = \varphi'(x_t/k_t) + \left[\varphi'(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi''(x_t/k_t)\right]$$

or

$$q_{(x/k)} = 2\varphi'(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi''(x_t/k_t) > 0,$$

Therefore, the only thing that firms need to observe in order to make financial decisions is q_t, the shadow price of investment. hence q is a sufficient statistic for investment, and investment is an increasing function of q. Therefore, when

$$\frac{x}{k} = h(q_t = 1) = 0$$
$$\frac{x_t}{k_t} = h(q_t > 1) > 0$$
$$\frac{x_t}{k_t} = h(q_t < 1) < 0$$

The intuition works as follows. The market cost of buying a machine is 1. If the firms know that they can buy a machine or capital at a price of 1, and get an revenue increase larger than 1 ($q > 1$) they will choose to. The installation costs will prevent any firm from doing large investment given that they increase the cost of investment. In a sense, this happens because the value of uninstalled capital is lower than the value of install capital because the uninstalled capital has to pay some adjustment costs. Install and uninstalled capital can be viewed as two different goods, and q_t is the relative price. In the neoclassical theory of investment they both have the same price $q_t = 1$. In this case the adjustment costs prevents it too be the case.

We need to determine the relative value q_t, or the shadow value of install capital. We can evaluate that from the last first-order conditions with respect to k_{t+1}

$$R = \frac{f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}(1-\delta)}{q_t}$$

The return from investing one unit of capital can be decomposed in three effects: the marginal product of capital MPK (implicit return from investment), the reduction in adjustment costs because it has a larger size, and finally, the capital gains captured by a larger value of future capital in terms of its future price per unit. As we can see, today's q_t depends on the future shadow value q_{t+1}.

$$q_t = R^{-1}[f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}(1-\delta)]$$

We can not solve for q_t until we know q_{t+1}. This is how future events will influence present investment decisions. In the neoclassical theory of investment marginal-q is always constant $q_t = 1$, or it declines an exponential rate $q_t = R^{-t}$. Setting $\delta = 0$ for simplicity, and then iterating on q_t, we have

$$\begin{aligned}
q_t &= R^{-1}[f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}] \\
q_{t+1} &= R^{-1}[f_k(k_{t+2}, l_{t+2}) + \left(\frac{x_{t+2}}{k_{t+2}}\right)^2 \varphi'(x_{t+2}/k_{t+2}) + q_{t+1}] \\
&\dots \\
q_{t+T} &= R^{-1}[f_k(k_{t+T}, l_{t+T}) + \left(\frac{x_{t+T}}{k_{t+T}}\right)^2 \varphi'(x_{t+T}/k_{t+T}) + q_{t+T+1}]
\end{aligned}$$

that is

$$q_t = \sum_{t=0}^{T} R^{-t}[f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1})] + R^{-t}q_{t+T+1}$$

where the transversality condition takes care of the last term. Finally,

$$q_t = \sum_{t=0}^{\infty} R^{-t}[f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1})].$$

The current value or price of installed capital is the present value of all future contributions of capital to profits/dividends. This is the direct marginal product of capital plus the reduction in adjustment costs associated to a larger firm size.

In this model anticipated changes in the marginal product of capital affect the current valuation of the current investment. Due to the presence of adjustment costs, firms will choose to smooth their investment decisions over time, instead of investing or disinvesting large amount upon the arrival of the shock.

5.2.2 Dynamics of Investment

The dynamics of the investment system can be characterized by three equations. The begin by discussing the steady state equilibrium, and then we focus on the dynamic path.

1. **Steady State:** In steady state there is no investment $x = 0$. Therefore, the model converges to the neoclassical theory of investment

$$\begin{aligned} w &= f_l(k, l), \\ q &= 1 + 0\varphi(0/k) + \left(\frac{0}{k}\right)\varphi'(0/k) = 1, \\ R &= f_k(k, l) + 1 - \delta, \end{aligned}$$

2. **Transitional Dynamics:** We have a system of two equations and two unknowns $\{q_t, k_{t+1}\}$

$$\begin{aligned} x_t &= k_{t+1} - (1-\delta)k_t \\ q_t &= 1 + x_t\varphi(x_t/k_t) + \left(\frac{x_t}{k_t}\right)\varphi'(x_t/k_t) \\ q_t R &= f_k(k_{t+1}, l_{t+1}) + \left(\frac{x_{t+1}}{k_{t+1}}\right)^2 \varphi'(x_{t+1}/k_{t+1}) + q_{t+1}(1-\delta) \end{aligned}$$

5.2.3 Marginal-Q vs Average-Q

The marginal-q theory of investment is very nice, but the main problem is that relies on the q a shadow price that it is not directly observable by firms. Therefore, it would be nice if we could replace this value by some observable parameter that we could use to test the theory.

Hayashi Theorem: If the production function and the total adjustment cost functions are homogeneous of degree one. Then, the shadow price is equivalent to the ratio of the market value of a firm divided by the replacement cost of capital.

Next, we want to proceed by showing that marginal-q is equal to average-q.

5.3 Modigliani-Miller Theorem Revisited

The Modigliani-Miller Theorem shows that under certain circumstances the total value of a firm is independent of the firm's financial structure. Consider the previous problem with an additional source of financing. Formally,

$$v(k_0) = \max_{\{b_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} R^{-t}(f(e_t) - rb_t - x_t),$$

$$\begin{aligned} s.t. \qquad k_{t+1} &= (1-\delta)(k_t + b_t) + x_t, \\ e_t &= k_t + b_t, \end{aligned}$$

Firms have an external b, and an internal source of financing k. The first-order conditions of the firm problem with respect to b_t, x_t, and k_{t+1} are given by

$$\begin{aligned} R^{-t}[f'(e_t) - r - q_t\delta] &= 0 \\ -R^{-t} + R^{-t}q_t &= 0 \\ -R^{-t}q_t + R^{-(t+1)}[f'(e_t) + (1-\delta)k_t] &= 0 \end{aligned}$$

Rearranging terms we have

$$\begin{aligned} q_t &= 1, \\ r &= f'(e_t) - \delta, \\ R &= f'(e_t) + 1 - \delta, \end{aligned}$$

The optimal plant size is determined by the right hand side of the expression.

$$e_t^* = k_t + b_t = f'(r, \delta)^{-1}$$

Theorem (Modigliani-Miller): *If there are no frictions in the capital markets, firms will be indifferent between internal k and external financing b.*

Firm attain an immediate size once they have access to capital markets. In particular, if $k_t > e_t^*$, then the firm chooses to lend funds in the credit market $b_t < 0$, and receives a return of $R =$

$r+\delta$. When, $k_t < e_t^*$ it does the opposite and borrows at resources $b_t > 0$. If the firm has an optimal size, $e_t^* = k_t$, then it does not external funds, $b_t = 0$. In this model firm achieve their optimal plant size in one period, and unless market conditions change, they will not change their size.

Example: Consider a production function of the form $f(e) = (k+b)^\theta$, We obtain,

$$\theta(k+b)^{\theta-1} = r$$

rearranging terms we have

$$e^* = k^* + b^* = \left(\frac{\theta}{r}\right)^{\frac{1}{1-\theta}}$$

The optimal plant size is determined by the right hand side of the expression. Firms will be indifferent between internal k and external financing b. Once we determine the optimal level, we can compute output

$$Q^* = f(e^*) = \left(\frac{\theta}{r}\right)^{\frac{\theta}{1-\theta}},$$

Once they obtain the optimal size, they stay there. With no additional financing costs firms attain the optimal size in one period.