Chapter 4

General Equilibrium under Uncertainty

4.1 Environment

4.2 Arrow-Debreu Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s)\}_{s=1}^S\}_{i=1}^I$ and state contingent prices $p = \{p_0, \{p_1(s)\}_{s=1}^S\}$ such that

1. Consumer i takes prices p as given and chooses the allocation x^i that solves

$$\max_{\{c_0^i, c_1^i(s)\}_{s=1}^S} U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s)),$$

$$s.t. \qquad p_0 c_0^i + \sum_{s=1}^S p_1(s) c_1^i(s) \leq p_0 y_0^i + \sum_{s=1}^S p_1(s) y_1^i(s),$$
$$c_0^i, c_1^i(s) \geq 0$$

2. Markets clear

$$\sum_{i=1}^I c_0^i \leq \sum_{i=1}^I y_0^i = Y_0,$$
$$\sum_{i=1}^I c_1^i(s) \leq \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s,$$

Example 1 (Equilibrium prices): Consider a two-state economy where consumers have preferences of the form $u(c) = \ln c$. The consumer problem becomes

$$\max_{\{c_1^i(s)\}_{s=1}^S} \sum_{s=1}^S \pi_s \log(c_1^i(s)),$$

$$s.t. \quad \sum_{s=1}^S p_1(s)c_1^i(s) \leq \sum_{s=1}^S p_1(s)y_1^i(s),$$
$$c_1^i(s) \geq 0$$

The first-order conditions of the consumer problem are given by

$$\frac{\pi_s}{c_1^i(s)} = \lambda p_1(s),$$

that implies an equal expected consumption expenditure across states. Formally,

$$\pi_s p_1(j)c_1^i(j) = \pi_j p_1(s)c_1^i(s)$$

Summing across all agents we have

$$\pi_s p_1(j)[c_1^1(j) + ... + c_1^I(j)] = \pi_j p_1(s)[c_1^1(s) + ... + c_1^I(s)]$$

or

$$\pi_s p_1(j)Y(j) = \pi_j p_1(s)Y(s)$$

Rearranging terms

$$\frac{p_1(j)}{p_1(s)} = \frac{\pi_j}{\pi_s}\frac{Y(s)}{Y(j)}$$

Example 2 (Idiosyncratic Uncertainty): Assume that the aggregate endowments are constant across states, formally, $Y(s) = Y(j)$ for s and j. Then, the equilibrium prices are given by

$$p_1(s) = \pi_s$$

Substituting in the consumers FOC we obtain perfect risk-sharing

$$c_1^i(s) = c_1^i(j)$$

for all i and s. From the households budget constraint we can compute the equilibrium consumption given by

$$c_1^i(s) = \sum_{s=1}^S \pi_s y_1^i(s),$$

Example 3 (Aggregate Uncertainty): Assume that the aggregate endowments fluctuate across states. In particular, consider an stochastic growth rate of the form $Y(s) = g(s)Y(1)$, where $g(s) > 0$. Then, the relative prices between a given state s and the normalized state 1 is

$$\frac{p_1(j)}{p_1(1)} = \frac{\pi_j}{\pi_1}\frac{g(s)Y(1)}{Y(1)},$$

or

$$\frac{p_1(j)}{p_1(1)} = \frac{\pi_j}{\pi_1}g(s)$$

4.3 Sequential Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s), b_1^i(s)\}_{s=1}^S\}_{i=1}^I$ and state contingent prices $q = \{q_1(s)\}_{s=1}^S$ such that

1. Consumer i takes prices q as given and chooses the allocation x^i that solves

$$\max_{\{c_0^i, \{c_1^i(s), b_1^i(s)\}_{s=1}^S\}} U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s)),$$

$$s.t. \quad c_0^i = y_0^i - \sum_{s=1}^S q_1(s)b_1^i(s),$$
$$c_1^i(s) \leq y_1^i(s) + b_1^i(s), \qquad \forall s,$$
$$c_0^i, c_1^i(s) \geq 0$$

2. Markets clear

$$\sum_{i=1}^I b_1^i(s) \leq 0, \qquad \forall s,$$
$$\sum_{i=1}^I c_0^i \leq \sum_{i=1}^I y_0^i = Y_0,$$
$$\sum_{i=1}^I c_1^i(s) \leq \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s,$$

Example 4 (Equilibrium prices): Consider the same example of the previous section where we ignore time 0 consumption, but not the asset markets. The consumer problem becomes

$$\max_{\{c_1^i(s), b_1^i(s)\}_{s=1}^S} \sum_{s=1}^S \pi_s \log(c_1^i(s)),$$

$$s.t. \quad \sum_{s=1}^{S} q_1(s) b_1^i(s) = y_0^i,$$
$$c_1^i(s) \leq y_0^i + b_1^i(s) \qquad \forall s,$$

Let λ and $\mu(s)$ denote the Lagrange multipliers of the budget constraints. The first-order conditions of the consumer problem are given by

$$\frac{\pi_s}{c_1^i(s)} = \mu^i(s),$$
$$q_1(s)\lambda^i = \mu^i(s)$$

Rearranging terms we obtain an equal expected consumption expenditure across states. Formally,

$$\pi_s q_1(j) c_1^i(j) = \pi_j q_1(s) c_1^i(s)$$

Summing across all agents we have

$$\pi_s q_1(j)[c_1^1(j) + ... + c_1^I(j)] = \pi_j q_1(s)[c_1^1(s) + ... + c_1^I(s)]$$

or

$$\frac{q_1(j)}{q_1(s)} = \frac{\pi_j}{\pi_s}\frac{Y(s)}{Y(j)}$$

Substituting into the FOC we obtain

$$\frac{Y(1)}{c_1^i(1)} = ... = \frac{Y(S)}{c_1^i(S)}$$

According to this expression, individual consumption is perfectly correlated with output, and aggregate consumption. In particular, if $\triangle Y(s)$ then $\triangle c_1^i(s)$ changes in the same magnitude, but is imperfectly correlated with individual income. The expected individual income determines the levels, but not the ratios across states of nature.

Example 5 (Idiosyncratic Uncertainty): We can use the previous example to compute the implied portfolios. Consider a specialized example with only two shocks. In the second period,

$$c_1^i(1) = c_1^i(2)$$

Now, we can compute the demand for Arrow securities combining time 0 and time 1 budget constraints,

$$q_1(1) b_1^i(1) + q_1(2) b_1^i(2) = 0$$

$$y_1^i(1) + b_1^i(1) = y_1^i(2) + b_1^i(2)$$

Rearranging terms we obtain,

$$\widehat{b}_1^i(1) = (1-\pi)[y^i(2) - y^i(1)]$$
$$\widehat{b}_1^i(2) = \pi[y^i(1) - y^i(2)]$$

4.4 Financial Markets

Definition: A market equilibrium is a state contingent consumption plan $x = \{x^i\}_{i=1}^I = \{c_0^i, \{c_1^i(s)\}_{s=1}^S, \{a_{1j}^i\}_{j=1}^J\}_{i=1}^I$ and state contingent prices $Q = \{Q_{1j}\}_{j=1}^J$ such that

1. Consumer i takes prices Q as given and chooses the allocation x^i that solves

$$\max_{x^i} U(c_0^i) + \beta \sum_{s=1}^{S} \pi_s U(c_1^i(s)),$$

$$s.t. \quad c_0^i = y_0^i - \sum_{j=1}^{J} Q_{1j} a_{1j}^i,$$
$$c_1^i(s) - y_1^i(s) \leq r_{11}(s) a_{11}^i(s) + ... + r_{1J}(s) a_{1J}^i(s), \qquad \forall s,$$
$$c_0^i, c_1^i(s) \geq 0$$

2. Markets clear

$$\sum_{i=1}^{I} a_{1j}^i(s) \leq 0, \qquad \forall j$$
$$\sum_{i=1}^{I} c_0^i \leq \sum_{i=1}^{I} y_0^i = Y_0,$$
$$\sum_{i=1}^{I} c_1^i(s) \leq \sum_{i=1}^{I} y_1^i(s) = Y(s), \qquad \forall s,$$

It is direct to show that all these economies are equivalent with complete markets. In particular, the sequential market structure is a special case of the financial markets economy, where the return of the asset in all the states but one are zero. The equilibrium prices have to satisfy the no-arbitrage property, so we can use the equilibrium allocations from the other economies to compute the price of any asset.

4.5 Pareto Efficiency

The social planner problem is the easiest way to find the optimal consumption allocations. Then, we can use the second welfare theorem to compute the implied equilibrium prices. Formally, the social planner solves

$$\max_x \sum_{i=1}^I \lambda_i \left[U(c_0^i) + \beta \sum_{s=1}^S \pi_s U(c_1^i(s))\right],$$

$$\begin{aligned} s.t. \quad & \sum_{i=1}^I c_0^i = \sum_{i=1}^I y_0^i = Y_0, \\ & \sum_{i=1}^I c_1^i(s) = \sum_{i=1}^I y_1^i(s) = Y(s), \qquad \forall s \end{aligned}$$

where $\sum_{i=1}^I \lambda_i = 1$. Let α be the Lagrange multiplier of the time 0 resource constraint, and $\mu(s)$ the multiplier of the resource constraint at time 1 in state s. The optimal allocation satisfies

$$\lambda_i U'(c_0^i) = \alpha$$
$$\lambda_i \beta \pi_s U'(c_1^i(s)) = \mu(s)$$

together with the $s+1$ resource constraints. Combining time 0 first-order conditions we have

$$\lambda_1 U'(c_0^1) = ... = \lambda_I U'(c_0^I)$$

In a symmetric equilibrium, individual consumption is a constant fraction of aggregate output. At time 1 we have a similar result.

$$\lambda_1 \beta \pi_s U'(c_1^1(s)) = ... = \lambda_I \beta \pi_s U'(c_1^I(s)),$$

or

$$U'(c_1^1(s)) = ... = U'(c_1^I(s))$$

The optimal allocation implies perfect insurance within the state. Individual consumption is only correlated with aggregate consumption, but not individual income shocks. That is

$$c_0^i = \frac{1}{I} Y_0,$$
$$c_1^i(s) = \frac{1}{I} Y_1(s)$$

Higher aggregate shocks lead to higher individual consumption levels, but the distribution does not matter. The determinate for asset market prices are not consumption levels, but ratios. Therefore, the ratio of consumption or marginal utility does not usually depend on the relative weight that the planner assigns.

Chapter 5

Competitive Equilibrium with Complete Markets

5.1 Environment

- Finite number of states $s \in S$
- $\pi(s'/s) = prob(s_{t+1} = s'/s_t = s)$ is a first-order Markov chain

  $\pi_0(s) = prob(s_0 = s)$ is the initial distribution

  $\pi(s^t)$ is a sequence of probability measures to achieve a particular history

  $$s^t = (s_t, s_{t-1}, s_{t-2}, ..., s_1, s_0)$$

  This probability can be computed via recursion

  $$\pi(s^t) = \pi(s_t/s_{t-1})\pi(s_{t-1}/s_{t-2})...\pi(s_1/s_0)\pi(s_0)$$

  This is the unconditional probability when s_0 has not been observed yet. When s_0 has been observed, we then have the conditional probability

  $$\pi(s^t/s_0) = \pi(s_t/s_{t-1})\pi(s_{t-1}/s_{t-2})...\pi(s_1/s_0)$$

  where $\pi(s^t/s_0) = \pi(s^t)\pi(s_0)$
- Finite number of agents $i \in I$
- Endowment for each household $y_t^i = y^i(s_t)$ is a time-invariant function that only depends on the the shock at time t.

- Endowments are publicly observable
- An allocation for agent i is defined as state contingent function $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$
- Preferences are represented by

$$U(c^i) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

or

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s^t))$$

where the utility function $u(\cdot)$ satisfies $u' > 0$, $u'' < 0$, C^2 and the Inada conditions $\lim_{t \to 0} u'(c) = +\infty$.

- An allocation is a list of sequence of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for all i. An allocation is said to be feasible if it satisfies

$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

Notice that consumption can depend on history, but the period income only depends on the realization of the shock.

5.2 Arrow-Debreu Markets

Household trade dated state-contingent claims to consumption. There is a complete set of claims. Trade takes place at $t = 0$ after the shock has been realized. The price of a claim on time t consumption contingent on history s^t is denoted by $p_t^0(s^t)$. The superscript 0 refers to the date at which trades occur, while the time subscript t refers to the date that deliveries are to be made. A price system is a sequence of functions $\{p_t^0(s^t)\}_{t=0}^{\infty}$.

A given household i solves

$$U(c^i) = \max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t/s_0) u(c_t^i(s^t))$$

$$s.to \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) y^i(s_t)$$

$$c_t^i(s^t) \geq 0$$

The single budget constraint implicitly assumes complete markets because it allows un restricted trade in all states of nature.

Definition (Competitive Equilibrium): A competitive equilibrium is a feasible allocation $\{c^i\}_{i=1}^{I} = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^{I}$ and a price system $\{p_t^0(s^t)\}_{t=0}^{\infty}$ such that the allocation solves each household problem.

Proposition: *The competitive equilibrium allocation is not history dependent.*

$$c_t^i(s^t) = c^i(s^t)$$

Proof: The first-order conditions of the consumer problem are given by

$$\beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \gamma^i p_t^0(s^t)$$

For two different consumers that face the same prices we have

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\gamma^i}{\gamma^j} \quad \forall t, s$$

The ratios of marginal utilities between pairs of agents is constant across time and states. In general, that will not imply constant consumption levels, but proportional. Latter we will show that in absence of aggregate uncertainty, consumption will be constant across time and states of nature.

The relative consumption is given by

$$c_t^i(s^t) = u'\left(u'(c_t^j(s^t))\frac{\gamma^i}{\gamma^j}\right)^{-1}$$

This fact comes from combining the first-order of the consumer problem with the resource constraint

$$\sum_{i=1}^{I} u'\left(u'(c_t^j(s^t))\frac{\gamma^i}{\gamma^j}\right)^{-1} = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

If the right-hand side does not depend on history, it only depends on the existing shock s_t. Therefore, the left-hand side does not depend on history either.■

The equilibrium price function is derived from the consumer first-order conditions

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{u'(c_t^i(s^t))}{\gamma^i}$$

At $t = 0$, we also have

$$p_0^0(s^0) = \frac{u'(c_0^i(s^0))}{\gamma^i}$$

or

$$p_t^0(s^t) = \beta\pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))}$$

where $p_0^0(s^0) = 1$. The ratio of expected marginal utilities gives the stochastic discount factor, and the return of the state-contingent claim is one unit of consumption. Therefore, the price has to be lower than one. Once we determine the consumption allocation, we can compute the equilibrium prices.

5.2.1 Risk Sharing

Economist are interested on the insurance properties of financial markets, and increase welfare. Consider a utility function of the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where $\sigma > 0$. The optimality condition of the market equilibrium implies

$$\frac{(c_t^i)^{-\sigma}}{(c_t^j)^{-\sigma}} = \frac{\gamma^i}{\gamma^j} \quad \forall t$$

or

$$c_t^j = c_t^j\left(\frac{\gamma^i}{\gamma^j}\right)^{\frac{1}{\sigma}} \quad \forall t$$

Complete markets assumption implies that consumption allocations to distinct agents are constant fractions of another. With this preferences, individual consumption is perfectly correlated with aggregate output or consumption, but is not correlated with individual income $y^i(s_t)$. The fraction of consumption that each agent receives is independent of s^t. Hence, the model exhibits an extensive cross-state and cross-time consumption smoothing.

5.2.2 No Aggregate Uncertainty

We consider an economy with two types of consumers, and a continuum of each type. The Markov process s_t takes place on the unit interval $s_t \in [0,1]$, such that $y^1(s_t) = s$ and $y^2(s_t) = 1 - s$. In the absence of aggregate

uncertainty, we know that the optimal choice implies perfect insurance $c_t^i = c_0^i$,

$$\beta\pi(s^t/s_0)\frac{u'(c_t^1(s^t))}{u'(c_0^1(s^0))} = p_t^0(s^t) = \beta\pi(s^t/s_0)\frac{u'(c_t^2(s^t))}{u'(c_0^2(s^0))}$$

That is in equilibrium, we have

$$\frac{u'(c_t^1(s^t))}{u'(c_0^1(s^0))} = \frac{u^2(c_t^1(s^t))}{u^2(c_0^1(s^0))}$$

From the first-order conditions of the consumer problem, we have

$$p_t^0(s^t) = \beta^t\pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{\gamma^i}$$

Substituting the first-order condition into the budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)\frac{u'(c_0^i)}{\gamma^i}[c_0^i - y^i(s_t)] = 0$$
$$\frac{u'(c_0^i)}{\gamma^i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)[c_0^i - y^i(s_t)] = 0$$

given that $u'(c_0^i)/\gamma^i \neq 0$, then it must be the case that

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)[c_0^i - y^i(s_t)] = 0$$

or

$$c_0^i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi(s^t/s_0) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)y^i(s_t)$$

where $\sum_{s^t}\pi(s^t/s_0) = 1$, so we have

$$c_0^i = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)y^i(s_t)$$

Finally, we check feasibility

$$\begin{aligned}c_0^1 + c_0^2 &= (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)y^1(s_t) + (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)y^2(s_t)\\ &= (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t/s_0)[\underbrace{y^1(s_t)}_{s} + \underbrace{y^2(s_t)}_{1-s}] = (1-\beta)\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi(s^t/s_0) = 1\end{aligned}$$

Using the optimal consumption levels, we can compute the implicit asset prices.

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)\frac{u'(c_t^i(s^t))}{\gamma^i}$$

with constant marginal utility, and using the usual normalization $\gamma^i = u'(c_t^i(s^t))$ and $p_0^0(s^0) = 1$. We obtain

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)$$

where remember that $\pi(s^t/s_0)$ is the conditional probability for this particular history when s_0 has been observed. An important feature is that prices do not depend on the idiosyncratic income shock. It only depends on the particular realization of a given history.

We can further specialize the example assuming a particular endowment process for both consumers. Formally, assume that $y^1 = (1, 0, 1, 0, ...)$ and $y^2 = (0, 1, 0, 1, ...)$. In this case $p_t^0(s^t) = \beta^t$. The implied consumption allocations for both consumers are given by

$$c_0^1 = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi(s^t/s_0) y^i(s_t) = (1-\beta)\sum_{t=0}^{\infty}\beta^{2t}1$$

or

$$c_0^1 = \frac{(1-\beta)}{(1-\beta^2)} = \frac{(1-\beta)}{(1-\beta)(1+\beta)} = \frac{1}{1-\beta}$$

and for the other consumer we have,

$$c_0^2 = \frac{\beta}{1-\beta}$$

The first-consumer is relatively wealthier because it receives the high shock on the first-period. That allows high to consume more because the present value of his/her future income is higher.

5.3 Contingent Claims or Sequential Markets Structure

In a seminal paper Arrow (1964) showed that one-period securities are enough to implement complete markets, as long as a new one-period market re-opens

for trading next period. In this economy, trade takes place at each date and state $t \geq 0$ using a set of contingent claims to one-period ahead state consumption. We prove that with a full array of these one period set of claims, the sequential market structure attains the same consumption allocation as the competitive equilibrium with Arrow-Debreu market structure.

In this economy, the sequential budget constraint is given by

$$c_t^i(s_t) + \sum_{s^{t+1}} Q(s_{t+1}/s_t) b_{t+1}^i(s_{t+1}) = y_t^i(s_t) + b_t^i(s_t) \quad \forall s$$

where $Q(s_{t+1}/s_t)$ denotes the price of one unit of consumption t time $t+1$ contingent on state s_{t+1} given that today is state s_t. We assume that this function does not depend on t. Notice that consumption only depends on the existing shock s_t, and does not depend on history. All the history for household i is summarized by its present wealth given by $b_t^i(s_t)$.

A given household i solves

Definition: A sequential equilibrium is an allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s_t), b_{t+1}^i(s_{t+1})\}_{t=0}^{\infty}\}_{i=1}^I$ and a price system $\{Q(s_{t+1}/s_t)\}_{t=0}^{\infty}$ such that

i) the allocation solves each household problem, and

$$U(c^i) = \max \sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi(s^t/s_0) u(c_t^i(s_t))$$

$$s.to \quad c_t^i(s_t) + \sum_{s^{t+1}} Q(s_{t+1}/s_t) b_{t+1}^i(s_{t+1}) = y_t^i(s_t) + b_t^i(s_t) \quad \forall s$$

$$c_t^i(s_t) \geq 0 \quad b_{t+1}^i(s_{t+1}) \geq -B(s_t) \quad \forall s$$

ii) Markets clear

$$\sum_{i=1}^{I} c_t^i(s_t) = \sum_{i=1}^{I} y^i(s_t) = Y(s_t)$$

$$\sum_{i=1}^{I} b_{t+1}^i(s_{t+1}) = 0$$

Proposition: If $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^I$ is the solution of the Arrow-Debreu competitive equilibrium, this allocation also is the solution of the sequential equilibrium.

Proof: From the first-order conditions of the sequential problem we have

$$Q(s_{t+1}/s_t) = \beta\pi(s_t/s_t)\frac{u'(c_{t+1}^i(s_{t+1}))}{u'(c_t^i(s_t))}$$

together with a transversality condition

$$\lim_{t\to\infty}\sum_{s^{t+1}} Q(s_{t+1}/s_t)b^i_{t+1}(s_{t+1}) = 0$$

That implies $b^i_{t+1}(s_{t+1}) > 0$ if $Q(s_{t+1}/s_t) = 0$, or $b^i_{t+1}(s^{t+1}) = 0$ if $Q(s_{t+1}/s_t) > 0$. The first-order conditions of the Arrow-Debreu equilibrium are

$$\frac{p^0_{t+1}(s^{t+1})}{p^0_t(s^t)} = \beta\pi(s^t/s_0)\frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}$$

That implies

$$Q(s_{t+1}/s_t) = \frac{p^0_{t+1}(s^{t+1})}{p^0_t(s^t)}$$

Finally, we need to show that the Arrow-Debreu consumption allocation satisfies the sequential budget constraint. In this case, we choose the initial level of wealth so the allocations are tha same $b^i_0 = 0$ for all i. Then, the porfolio decisions should be the same in both economies. To show it we need to add up all the budget constraints across states of nature s and across time t, basically across all histories s^t. We start at time $t = 0$

$$\begin{array}{lll} t=0 & s=1 & p^0_0(1)[c^i_0(1) - y^i_0(1)] = p^0_0(1)b^i_0(1) - \sum_{s^1} p^0_0(1)Q(s_1/1)b^i_1(1) \\ \ldots & \ldots & \ldots \\ t=1 & s=S & p^0_0(S)[c^i_0(S) - y^i_0(S)] = p^0_0(S)b^i_0(S) - \sum_{s^1} p^0_0(S)Q(s_1/S)b^i_1(S) \end{array}$$

If we add them up we have

$$\sum_{s^0} p^0_0(s^0)[c^i_0(s^0) - y^i_0(s_0)] = \sum_{s^0} p^0_0(s^0)b^i_0(s_0) - \sum_{s^1}\underbrace{p^0_0(s^1)Q(s_1/s_0)}_{p^0_1(s^1)}b^i_1(s_1)$$

For the next periods we have

$$\sum_{s^1} p^0_1(s^1)[c^i_1(s^1) - y^i_1(s_1)] = \sum_{s^1} p^0_1(s^1)b^i_1(s_1) - \sum_{s^2} p^0_2(s^2)b^i_2(s_2)$$

$$\ldots$$

$$\sum_{s^{t-1}} p^0_{t-1}(s^{t-1})[c^i_{t-1}(s^{t-1}) - y^i_{t-1}(s_{t-1})] = \sum_{s^{t-1}} p^0_{t-1}(s^{t-1})b^i_{t-1}(s_{t-1}) - \sum_{s^t} p^0_t(s^t)b^i_t(s_t)$$

$$\sum_{s^t} p^0_t(s^t)[c^i_t(s^t) - y^i_t(s_t)] = \sum_{s^t} p^0_t(s^t)b^i_t(s_t) - \sum_{s^{t+1}} p^0_{t+1}(s^{t+1})b^i_{t+1}(s_{t+1})$$

If we add them all up,

$$\sum_{s^0} p^0_0(s^0)[c^i_0(s^0) - y^i_0(s_0)] + \ldots + \sum_{s^t} p^0_t(s^t)[c^i_t(s^t) - y^i_t(s_t)] = \sum_{s^{t+1}} p^0_{t+1}(s^{t+1})b^i_{t+1}(s_{t+1})$$

if we take the limit in both sides de have

$$\sum_{t=0}^{\infty}\sum_{s^t} p^0_t(s^t)[c^i_t(s^t) - y^i_t(s_t)] = \lim_{t\to\infty}\sum_{s^{t+1}} p^0_{t+1}(s^{t+1})b^i_{t+1}(s_{t+1}) = 0$$

5.4 Pareto Efficient Allocations

It is useful to have a welfare measure to compare the outcomes of different trading mechanism. We focus on Pareto efficient.

Definition (Pareto Efficiency): An allocation $\{c^i\}^I_{i=1} = \{\{c^i_t(s_t)\}^\infty_{t=0}\}^I_{i=1}$ is said to be Pareto efficient, if there not exists another feasible allocation $\{\tilde{c}^i\}^I_{i=1}$ such that

$$\begin{array}{ll} U(\tilde{c}^i) \geq U(c^i) & \forall i \\ U(\tilde{c}^i) > U(c^i) & \text{some } i \end{array}$$

The set of Pareto efficient allocation can be calculated by computing the so calles social planner problem. Consider a social planner that has to allocate resources among a large number of households. We assume that each consumer receives a time invariant discount rate $\lambda^i \in (0,1)$, and $\sum_{i=1}^I \lambda^i = 1$. The benevolent planner maximizes

$$U(c^1, ..., c^I) = \max_{\{c^i_t(s^t)\}^I_{i=1}} \sum_{i=1}^I \lambda^i \sum_{t=0}^\infty \sum_{s^t} \beta^t \pi(s^t/s_0)u(c^i_t(s^t))$$

$$\begin{array}{ll} s.to & \sum_{i=1}^I c^i_t(s^t) = \sum_{i=1}^I y^i(s_t) = Y(s_t) \\ & c^i_t(s^t) \geq 0 \end{array}$$

Let μ denote the Lagrange multiplier of the resource constraint. The first-order conditions for a given consumer i with respect to $c^i_t(s^t)$ are

$$\lambda^i \beta^t \pi(s^t/s_0)u'(c^i_t(s^t)) = \mu$$

Notice that marginal utility of consumption only depends on the aggregate variables, not on the individual income shock $y^i(s_t)$. Formally,

$$u'(c_t^i(s^t)) = \frac{\mu}{\lambda^i \beta^t \pi(s^t/s_0)}$$

or

$$c_t^i(s^t) = u\left(\frac{\mu}{\lambda^i \beta^t \pi(s^t/s_0)}\right)^{-1}.$$

For two different consumers i and j we have

$$\frac{\lambda^i \beta^t \pi(s^t/s_0) u'(c_t^i(s^t))}{\lambda^j \beta^t \pi(s^t/s_0) u'(c_t^j(s^t))} = 1$$

or

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\lambda^j}{\lambda^i}$$

Clearly, the allocation of consumption across households depends on the relative weight that the social planner assigns to each household. In particular, if $\lambda^i > \lambda^j$ then $u'(c_t^i(s^t)) < u'(c_t^j(s^t))$, and $c_t^i(s^t) > c_t^j(s^t)$. The agent with higher weight receives more consumption. In a symmetric allocation $\lambda^i = \lambda^j$ all agents receive the same allocation, $c_t^i(s^t) = \alpha Y(s_t)$, where $\alpha = 1/I$. Individual consumption only depends on the aggregate shock, not on the idiosyncratic labor income shock. Finally, we can replace the optima consumption levels on the first-order conditions

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{u'(\alpha Y(s_t))}{u'(\alpha Y(s_t))} = \frac{\lambda^j}{\lambda^i}$$

and obtain $\lambda^j = \lambda^i$ that both agents need to have the same initial wealth to achieve the symmetric allocation. If all agents do not have the same initial wealth, it is necessary to implement lump-sum taxes to achieve this allocation.

5.5 First and Second Welfare Theorems

First, we want to prove the so called first-welfare theorem. The theorem highlights some of the nice welfare properties of complete markets economies.

Proposition (First-welfare theorem): An equilibrium allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^\infty\}_{i=1}^I$ in the market economy is Pareto efficient.

Proof: Suppose the contrary, then there exists another feasible allocation $\{\widetilde{c}^i\}_{i=1}^I$ that Pareto dominates the equilibrium allocation. At the equilibrium prices $\{p_t^0(s^t)\}_{t=0}^\infty$, this allocation has to cost strictly more than the endowment for the individual that can be improved. Otherwise this agent is not maximizing utility. That is

$$\sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) \widetilde{c}_t^i(s^t) > \sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) y^i(s_t)$$

for the other consumers this constraint is satisfied with equality. If we add up all the constraints we find

$$\sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) \sum_{i=1}^I \widetilde{c}_t^i(s^t) > \sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) \sum_{i=1}^I y^i(s_t)$$

$$\sum_{t=0}^\infty \sum_{s^t} p_t^0(s^t) \left(\sum_{i=1}^I \widetilde{c}_t^i(s^t) - Y(s_t)\right) > 0$$

given that $p_t^0(s^t) > 0$ for all t and s, the alternative allocation $\{\widetilde{c}^i\}_{i=1}^I$ is not feasible. That contradicts the assumption of Pareto efficient allocations. Clearly, there exists better allocations but there are not feasible.■

Proposition (Second-welfare theorem): An allocation $\{c^i\}_{i=1}^I = \{\{c_t^i(s^t)\}_{t=0}^\infty\}_{i=1}^I$ is Pareto efficient, there exists a price system that supports this allocation as a market equilibrium.

Proof: If we compare the first-order conditions of the social planner

$$\lambda^i \beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \mu$$

with the competitive equilibrium from the previous section

$$\beta^t \pi(s^t/s_0) u'(c_t^i(s^t)) = \gamma^i p_t^0(s^t)$$

It is clear, that both economies will deliver the same allocations if $\mu/\lambda^i = \gamma^i p_t^0(s^t)$. There exist a vector of relative weight $\{\lambda^i\}_{i=1}^I$, such that the solution of both economies is the same given the initial distribution of entitlements. In particular, we can use the social planner allocations two compute the optimal consumption, and the implied equilibrium price system.

For the symmetric case, that is $\lambda^1 = ... = \lambda^I$, where $c_t^i(s^t) = \alpha Y(s_t)$ for all i and $\alpha = 1/I$ represents the individual share on aggregate output.

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{u'(\alpha Y(s_t))}{u'(\alpha Y(s_0))}$$

The price of state-contingent claims depends on the comovements of output.■

Assume that the changes in aggregate output across time and states is given by $Y(s_t) = g(s_t)Y(s_0)$, where you can think of $Y(s_0)$ as the average level of output. We can rewrite this equation as

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{u'(\alpha g(s_t) Y(s_0))}{u'(\alpha Y(s_0))}$$

In the absence of aggregate uncertainty $g(s_t) = 1$ for all s and t. Then, the equilibrium prices are given by

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0)$$

We obtain the same pricing that with risk-neutral preferences $u(c) = c$. In the presence of aggregate uncertainty and isolastic preferences $u'(c) = c^{-\gamma}$,

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) \frac{(\alpha g(s_t) Y(s_0))^{-\gamma}}{(\alpha Y(s_0))^{-\gamma}}$$

or

$$p_t^0(s^t) = \beta^t \pi(s^t/s_0) g(s_t)^{-\gamma}$$

The price of consumption goods is lower in states with high output growth, and higher in states with low ouput growth. Agents with high endowments in periods with low output are relatively wealthier.

One way to test the model is to use estimate a process for consumption growth, and see whether the implied equilibrium prices satisfy the some properties observed in the data.

The advantatge of the second welfare theorem, is that we can use the social planner problem to compute the optimal allocations, and the used them to derive the equilibrium prices. Notice that the equilibrium prices do not depend on the social planner weight, because they depend on the ratio of marginal utilities, and this ratio is unaffected by the weight. We will exploit this result to solve Lucas model of asset prices.

5.6 Lucas Model of Asset Prices

The two previous specifications do not specify the market structure that yields a constant interest rate for example. Lucas asset pricing model uses a simple exchange economy to determine the pricing function. The economy considers a large number of identical agents which receive no labor income. We consider and economy populated by a large number of identical households solving

$$\max_{\{c_t, s_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.a. \qquad c_t + p_t s_{t+1} = s_t(p_t + d_t)$$
$$s_{t+1} \geq -B$$

where B is a large positive constant that never binds but prevents Ponzi schemes. Notice that we have set $y_t = 0$ in all t. The only durable good is a set of "trees" which are equal in number to the number of people in the economy. At each period t, each tree yiels a fruit or dividend in the amout d_t to its owner. We assume that the dividend is nonstorable, but the tree is perfectly durable. The solution of this problem yields

$$p_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}(p_{t+1} + d_{t+1})]$$

together with a transversality condition $\lim_{j \to \infty} \beta^j u'(c_{t+j}) p_{t+j} = 0$. This condition says that in the limit consumer will not hold assets if the price is positive, or will hold positive amounts if the price is zero.

The competitve equilibrium consumption allocation of this economy can be readily be computed once we notice that the economy can be treated as autarkic. Because preference and endowment patterns are the same across individuals, there can be no gains from trade. In equilibrium it must be the case that $c_t = d_t$ because the utility function $u(\cdot)$ is strictly increasing (that means no satiation), and the dividend is the only source of consumption goods. We can deal with a representative consumer directly.

In equilibrium, prices have to be such that markets clear. That means that the total amout of borrowing in the economy is zero, and the share holdings has to be one, $s_t = 1$. Sustituting the equilibrium conditions in the

Euler equation, and using the law of iterated expectations we conclude that the price of a share must satisfy

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j} + E_t \lim_{j\to\infty} \beta^j u'(d_{t+j}) p_{t+j}$$

The transversality condition of the consumer problem rules out solutions that include a bubble term. If the last term where positive $E_t \lim_{j\to\infty} \beta^j u'(d_{t+j}) p_{t+j} > 0$, the marginal utility of selling shares excess the marginal utility of holding assets and consume the expected flow $p_t u'(d_t) > E_t \sum_{j=1}^{\infty} \beta^j u'(d_{t+j}) d_{t+j}$. Consequently, all households would sell share to increase their consumption, and as a result the price of a share will fall. We have a similar argument if the additional term is negative. There in equilibrium it must be the case that

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}$$

or

$$p_t = E_t \sum_{j=1}^{\infty} m_{t+j} d_{t+j}$$

where $m_{t+j} = \beta^j u'(d_{t+j})/u'(d_t)$ represents the stochastic discount factor. This equation is a generalization of the random walk theory of stock prices, in which the share price is an expected discounted stream of dividends but with a time-varing and stochastic discount rate m_{t+j} that is different from one as in the previous case. We can decompose the price of the asset in two terms: the discounted value of the consumption flow and the correcttion term for risk. Formally,

$$p_t = \sum_{j=1}^{\infty} \left(E_t(m_{t+j}) E_t(d_{t+j}) + cov(m_{t+j}, d_{t+j}) \right)$$

with risk neutral preferences we have that $m_{t+j} = \beta^i$, and with perfect insurance we have that $u'(c_t) = E_t u'(c_{t+1})$ and $cov(m_{t+j}, d_{t+j}) = 0$. So the price of an asset is the discounted sum of future dividends

$$p_t = \sum_{j=1}^{\infty} E_t(d_{t+j}),$$

In general, that will not be the case and the asset will be adjusted by the premium factor. Given that $c_t = d_t$, it must be the case that when there is a good shock $\triangle d_{t+j} \to \triangle c_{t+j} \to \nabla u'(c_{t+j}) \to \nabla m_{t+j} = u'(c_{t+j})/u'(c_t)$. Then, $cov(m_{t+j}, d_{t+j}) < 0$ so we have

$$p_t < \sum_{j=1}^{\infty} E_t(m_{t+j}) E_t(d_{t+j})$$

if we normalize $E_t(m_{t+j}) = 1$, we have that the price of a risky asset should be lower than the expected discountes stream of its dividends. That also means that the return of that asset is higher because otherwise households will not buy this asset.

This version of the Lucas model has been used to generate allocations and price of assets, and compare them with the data. These asset pricing models can be constructed as follows:

1. We describe the preferences, technology and endowments. Given a particular market structure where agents are allowed to buy and sell assets, we solve for the equilibrium consumption allocations.

2. Sometimes there exists a planning problem whose solution equals the competitive equilibrium. Therefore, we can equate the consumption that appears on the Euler equation, and compute the implied asset price at time t as a function of the state of the economy at t.

In our endowment economy, a benevolent social planner would solve

$$\max_{\{c_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.to \qquad c_t \le d_t, \quad c_t \ge 0$$

After substituting the consumption allocations into the priceing equations we derive the standad equation for a price of a share

$$u'(d_t) p_t = E_t[\beta u'(d_{t+1})(p_{t+1} + d_{t+1})]$$

or

$$p_t = E_t[\beta m_{t+1} x_{t+1}]$$

where $m_{t+1} = u'(c_{t+1})/u'(c_t)$ and $x_{t+1} = (p_{t+1} + d_{t+1})$. Next, we want to study some special cases

Example 1: Logarithmic utility function

Consider a utility function of the form $u(c_t) = \ln c_t$, where $u'(c_t) = c_t^{-1}$. If we replace this expression in the pricing equation we obtain

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_t}{d_{t+j}} d_{t+j}$$

rearranging terms

$$p_t = d_t E_t \sum_{j=1}^{\infty} \beta^j$$

or

$$p_t = \frac{1}{1-\beta} d_t$$

This equation is an example of an asset-pricing function that maps the state of the economy at t, d_t into the priceof a capital asset at t. In particular, the dividend at time t is all the information requiered to predict the price. In this particular case $m_{t+j} = d_t/d_{t+j}$ and does not necessarily need to be one. The price is a linear function of the aggregate state of the economy. This is a property that we will exploid in detail in this chapter.

Example 2: Risk neutrality or perfect insurance

If the utility function is linear in consumption, $u(c_t) = c_t$, then the ratio of marginal utilities is constant. That is $m_{t+j} = u'(c_{t+j})/u'(c_t) = 1$. Then, the price of a share at time t is

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t d_{t+j}$$

in this case the price of the share depends on the stochastic properties of the dividend process.

- **First-order autoregressive process:** If we assume that dividends follow a first-order autoregressive process

$$d_{t+1} = \alpha + \rho d_t + \varepsilon_{t+1}$$

where ε_{t+1} is white noise, that is $E(\varepsilon_{t+1}) = 0$. Then, the expected value of the dividend is at $t+1$ is

$$\begin{aligned}
E_t[d_{t+1}] = &\ \alpha + \rho d_t \\
E_t[d_{t+2}] = &\ E[\alpha + \rho d_{t+1} + \varepsilon_{t+2}] = \alpha + \rho E[d_{t+1}] = \alpha + \rho(\alpha + \rho d_t) \\
E_t[d_{t+3}] = &\ \alpha + \rho E[d_{t+2}] = \alpha(1 + \rho + \rho^2) + \rho^3 d_t \\
\vdots &\ \qquad \vdots \\
E_t[d_{t+k}] = &\ \alpha(1 + \rho + \rho^2 + \ldots + \rho^{k-1}) + \rho^k d_t
\end{aligned}$$

or

$$p_t = \sum_{j=1}^{\infty} \beta^j [\alpha(1 + \rho + \rho^2 + \ldots + \rho^{j-1}) + \rho^j d_t]$$

$$p_t = \alpha \sum_{j=1}^{\infty} \beta^j \sum_{j=1}^{\infty} \rho^j + \sum_{j=1}^{\infty} (\beta\rho)^j d_t$$

- **I.i.d. shocks:** If we assume that the process is independent and identically distributed according to $\varepsilon \sim N(0, \sigma^2)$, then

$$d_t = \alpha + \varepsilon_{t+1}$$

the price of the dividend flow is given by

$$p_t = \sum_{j=1}^{\infty} \beta^j E_t(\alpha + \varepsilon_{t+1}) = \frac{\alpha}{1-\beta},$$

In this case prices are set to the mean value of the dividend process.

5.6.1 Equivalent Allocations

Next, we show that the Lucas model, or asset structure yields the same allocations that the Arro-Debreu markets

Proposition: If $\{c^i\}_{i=1}^I = \{\{c_t^i(z_t), s_{t+1}^i(z_t)\}_{t=0}^{\infty}\}_{i=1}^I$ is the solution of the Lucas model, then, the consumption and asset allocation also is the solution of the Arrow-Debreu competitive equilibrium.

Proof: We start with the sequential Lucas constraint for a particular realization of the dividend shock.

$$c_t - y_t = s_t(q_t + d_t) - q_t s_{t+1}$$

we define the Arrow-Debreu prices as

$$\frac{p_t}{p_{t+1}} = \frac{q_{t+1} + d_{t+1}}{q_t}$$

or $p_t q_t = p_{t+1}(q_{t+1}+d_{t+1})$. Now we multiply each sequential budget constraint by is respective price $p_t, p_{t+1}, p_{t+2}, ...$ Formally, we have

$$\begin{aligned} p_t[c_t - y_t] &= p_t(q_t + d_t)s_t - p_t q_t s_{t+1} \\ p_{t+1}[c_{t+1} - y_{t+1}] &= p_{t+1}(q_{t+1} + d_{t+1})s_{t+1} - p_{t+1}q_{t+1}s_{t+2} \\ p_{t+2}[c_{t+2} - y_{t+2}] &= p_{t+2}(q_{t+2} + d_{t+2})s_{t+2} - p_{t+2}q_{t+2}s_{t+3} \\ &... \end{aligned}$$

If we add them up

$$\sum_{t=1}^{\infty} p_t[c_t - y_t] = p_t(q_t + d_t)s_t - s_{t+1}\underbrace{[p_t q_t - p_{t+1}(q_{t+1} + d_{t+1})]}_{=0} - s_{t+2}\underbrace{[p_{t+1}q_{t+1} - p_{t+2}(q_{t+2} + d_{t+2})]}_{=0} + ...$$

Now we need to solve for $p_0(q_0 + d_0)s_0$

$$p_0 q_0 s_0 + p_0 d_0 s_0$$

where $p_0 = p_1(q_1 + d_1)/q_0$

$$\frac{p_1(q_1 + d_1)}{q_0} q_0 s_0 + p_0 d_0 s_0 = p_1(q_1 + d_1)s_0 + p_0 d_0 s_0 = p_1 q_1 s_0 + (p_1 d_1 + p_0 d_0)s_0$$

that is

$$\sum_{t=1}^{\infty} p_t d_t s_0$$

Combining all together we have

$$\sum_{t=1}^{\infty} p_t[c_t - y_t] = \sum_{t=1}^{\infty} p_t d_t s_0$$

Now, we just need to add-across states of nature

$$\sum_{t=1}^{\infty} \sum_{s^t} p_t^0(s^t)[c_t(s^t) - y_t(s_t)] = \sum_{t=1}^{\infty} \sum_{s^t} p_t^0(s^t) d_t(s_t) s_0$$

The model is equivalent to the Arrow-Debreu complete markets model, where agents receive an endowment or initial share on the tree, s_0. The price of the shares can be used to price the equivalent state $t - 0$ contingent claims.

5.6.2 The Random Walk Theory of Consumption

The next two theories emerge from studying marginal conditions for the consumer's problem and imposing some restrictions upon them. As we will see latter on, it is possible to describe simple market equilibrium setups that deliver these restrictions.

First, we analyze the randome walk theory of consumption formulated by Hall (1978). According to Hall the evolution of future consumption follows a random walk, and no variable in the information set can be used to predict it.[1] This theory is based on the stochastic Euler equation derived in the previous section. Formally,

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_{t+1}]$$

Hall assumes that in the economy there exists a risk-free rate asses with constant return $R_t = R > 1$. Under this assumption we can rewrite the Euler equation as

$$u'(c_t) = \beta E_t[u'(c_{t+1})]R$$

or

$$E_t[u'(c_{t+1})] = (\beta R)^{-1} u'(c_t)$$

This equation shows that the marginal utility of consumption follows a univariate first-order Markov process and that no other variables in the information set help to predict. We can rewrite the previous expression to include an error term on it. Formally,

$$E_t[u'(c_{t+1})] = (\beta R)^{-1} u'(c_t) + \varepsilon_{t+1}$$

We can further specialize the problem if we assume some particular preferences.

Example 1: Quadratic utility function

Cosnider a simple quadratic utility function given by

$$u(c_t) = a + bc_t + dc_t^2,$$

[1] Put in prespectivr this theory

$$C_t = f(Y_t)$$

and discuss the PIH in constrast with standard Keynesian theory.