Chapter 7

Competitive Equilibrium with Incomplete Markets

7.1 Environment

We consider an economy with discrete time periods $t = 0, 1, ...$ There are two types of consumers $i = 1, 2$ and a continuum of each type. We denote by c_t^i the single consumption good consumed each period, and $(c_0^i, c_1^i, ...) \in l_\infty^{++}$ is the infinite vector of consumption. Individual preferences are given by

$$U(c_0^i, c_1^i, ...) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

where the utility function satisfies $u' > 0$, $u'' < 0$, Inada conditions and the individual discount rate is $\beta \in (0, 1)$.

We assume that households have two forms of income/or capital: human (labor) and physical (trees or land). Let w_t be services of human capital, where $w_t \in (\omega^g, \omega^b)$ good and bad endowment, $\omega^g > \omega^b$. We assume that productivity fluctuates according to the transition matrix

$$\Pi_{w'/w} = \begin{bmatrix} \pi_{gg} & \pi_{bg} \\ \pi_{gb} & \pi_{bb} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Consequently, productivity alternates, $w_t = \omega^g \Longrightarrow w_{t+1} = \omega^b$. With respect to the other form of income, let d_t be the return on physical capital, where s_t^i denotes the share on the capital stock at t. The aggregate resource constraint is

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega$$

7.2 Equilibrium Prices in a Liquidity Constrained Economy

Next, we define the notion of market equilibrium in a liquidity constraint economy. Then, we focus in the solution of a symmetric steady state allocation. We distinct the solution where the liquidity constraint binds, and one where it does not bind.

Definition: *A market equilibrium in this economy is an allocation* $\{\{c_t^i, \theta_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ *and a sequence of prices* $\{q_t, r_t\}_{t=0}^{\infty}$, *such that*

- *Consumers solve*

$$\max (1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t^i)$$

$$\begin{aligned} s.t. \qquad c_t^i + q_t s_{t+1}^i &\leq w_t^i + (q_t+d)s_t^i \qquad \forall t \\ s_t^i \geq 0, &\qquad s_0^i \text{ given} \end{aligned}$$

- *Goods and financial markets clear*

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \qquad \forall t$$

$$s_t^1 + s_t^2 \leq 1 \qquad \forall t$$

We focus the attention on the steady state of both economies. We want to compute the decision rules for both shocks.

$$c_t^i = \begin{cases} c^g \text{ if } w_t^i = \omega^g \\ c^b \text{ if } w_t^i = \omega^b \end{cases}$$

Because $c^g + c^b = \omega$ we can characterize the symmetric steady state by a single number c^g, that is $c^b = \omega - c^g$. The analysis uses the first-order conditions to compare the consumption paths is both economies. The Euler equation of this problem is given by

$$\frac{u'(c_t^i)}{\beta u'(c_{t+1}^i)} \geq \frac{q_{t+1}+d}{q_t} \qquad (= 0 \text{ if } s^i > 0)$$

The consumer with ω^g can by as much capital for the consumer with ω^b, that is constraint by $s^i \geq 0$.

7.2.1 Liquidity Constraint does not bind

If the constraint does not bind, one possible equilibrium is a symmetric equilibrium. In particular a symmetric allocation need to satisfy the aggregate resource constraint $c^* = c^g = c^b = (\omega^g + \omega^b + d)/2 = \omega/2$. The Euler equation for a symmetric equilibrium is also satisfied. Formally,

$$\frac{u'(c^g)}{\beta u'(c^b)} = \frac{q+d}{q} = \frac{u'(c^b)}{\beta u'(c^g)}$$

or

$$\frac{1}{\beta} = \frac{q+d}{q}$$

Then, the equilibrium prices satisfies

$$p^* = \frac{\beta}{1-\beta}d$$

The allocation in a symmetric equilibrium satisfy

- Consumer first-order conditions,
- Aggregate resource constraint,
- Consumer budget constraint,
- Financial markets should clear

From the aggregate resource constraint we have

$$[c^g - \omega^g] + [c^b - \omega^b] = d$$

substituting the budget constraint for each household

$$[(p+d)s^b - ps^g] + [(p+d)s^g - ps^b] = d$$

rearranging terms we have

$$(p+d)(s^b + s^g) - p(s^b + s^g) = d.$$

When the financial markets clear $s^b + s^g = 1$, then, the aggregate resource constraint as well as the consumer budget constraint are satisfied. Now, we can compute the steady state trade associated to the optimal consumption level. Formally,

$$\frac{\omega}{2} - \omega^g = (p+d)s^b - ps^g,$$
$$\omega^b - \frac{\omega}{2} = (p+d)s^g - ps^b,$$

We can solve for the optimal share distribution by solving a linear system of equations. That is

$$\begin{bmatrix} p+d & -p \\ -p & p+d \end{bmatrix} \begin{bmatrix} s^b \\ s^g \end{bmatrix} = \begin{bmatrix} \frac{\omega}{2} - \omega^g \\ \omega^b - \frac{\omega}{2} \end{bmatrix}.$$

Example: Consider an economy where $\omega^g = 8$ and $\omega^b = 1$, where $d = 1$ and $\beta = 0.9$. If the utility function is $u(c) = \ln c$. The symmetric equilibrium allocation implies

$$\omega = \omega^g + \omega^b + d = 10$$

Then, we have $c^* = 10/2 = 5$. The equilibrium prices for shares in the tree are given by

$$p = \frac{0.9}{1-0.9}1 = 9,$$

Now, we can compute the portfolio holdings of each individual

$$\begin{bmatrix} s^b \\ s^g \end{bmatrix} = \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

The asset shares for the good and bad state are $s^b = 0.648$ and $s^g = 0.316$. As we can clearly, see in the example the liquidity constraint does not bind.

7.2.2 Liquidity Constraint does bind

The symmetric of the shocks implies for consumer with the good shock

$$\frac{u'(c^g)}{\beta u'(c^b)} = \frac{q+d}{q} \qquad s^g = 1$$

and for consumer with the bad shock.

$$\frac{u'(c^b)}{\beta u'(c^g)} > \frac{q+d}{q} \qquad s^b = 0$$

Then, MRS are not equal across consumers

$$\frac{q+d}{q} = \frac{u'(c^g)}{\beta u'(\omega - c^g)} < \frac{u'(\omega - c^g)}{\beta u'(c^g)}$$

The MRS are not equal across consumers. The equilibrium prices are determined by the individual that it is not borrowing constraint. The first-order conditions for the constraint are not important to compute the equilibrium. In this economy, the constraint agent is the individual with the bad income shock that would like to borrow to insure consumption fluctuations. Then, from the Euler equations we obtain

$$\widehat{s}^g = 1 \text{ and } \widehat{s}^b = 0.$$

From the consumer budget constraint with the good shock

$$c^g + q = \omega^g \Rightarrow q = \omega^g - c^g$$

From the consumer budget constraint with the bad shock

$$c^b = \omega^b + (q+d) \Rightarrow q + d = c^b - \omega^b$$

When the borrowing constraint binds, we have to different ways to compute the equilibrium allocation and prices in the economy. We have to solve this functional equation

$$qu'(c^g) = \beta u'(\omega - c^g)(q+d)$$

together with the budget constraints.

- **Compute equilibrium allocation:** We proceed by replacing the budget constraint into the FOC of the unconstrained consumer

$$\frac{u'(c^g)}{\beta u'(c^b)} = \frac{q+d}{q} = \frac{c^b - \omega^b}{\omega^g - c^g}$$

using feasibility $c^b + c^b = \omega$,

$$\frac{u'(c^g)}{\beta u'(\omega - c^g)} = \frac{(\omega - c^g - \omega^b)}{(\omega^g - c^g)}$$

Rearranging terms we obtain,

$$F^L(c^g) = u'(c^g)(\omega^g - c^g) - \beta u'(\omega - c^g)(\omega - c^g - \omega^b)$$

The equilibrium solve the functional equation on c^g. Next, we derive some properties of the equilibrium for this economy.

Proposition 1: The behavior of the economy can be characterized by the sign of the $F^L(c^g)$ function: 1) If the borrowing constraint binds, $F^L(c^g) = 0$, then $c^g > c^b$. 2) If the borrowing constraint does not bind, $F^L(c^g) \geq 0$, then $c^g = c^b$.

- **Compute equilibrium prices:** We proceed in a similar fashion, but we substitute allocations into the Euler equation to derive the equilibrium prices. From the consumer budget constraint with the good shock

  $$c^g = \omega^g - q,$$

  and from the consumer budget constraint with the bad shock

  $$c^b = \omega^b + (q + d)$$

  Then,

  $$\frac{u'(\omega^g - q)}{\beta u'(\omega^b + (q+d))} = \frac{q+d}{q}$$

  Rearranging terms we obtain,

  $$F^L(q) = \frac{u'(\omega^g - q)}{u'(\omega^b + (q+d))} - \beta\frac{(q+d)}{q}$$

The equilibrium solve the functional equation on q. Next, we derive some properties of the equilibrium for this economy.

Proposition 2: The behavior of the economy can be characterized by the sign of the $F^L(q)$ function: 1) If the borrowing constraint binds, $F^L(q) = 0$, then $c^g > c^b$. 2) If the borrowing constraint does not bind, $F^L(q) \geq 0$, then $c^g = c^b$.

7.2.3 Short Sales Constrained

In the previous model we assumed, $s_t^i \geq 0$. Now, we want to relax this assumption by setting $s_t^i \geq -A$. In the borrowing constraint case

$$s^b = -A$$
$$s^g = 1 + A$$

Substituting this decisions in the households budget constraint we find,

$$q = \frac{\omega^g - c^g - Ad}{(1+2A)}$$
$$q + d = \frac{c^b - \omega^b - d(1+A)}{(1+2A)}$$

substituting into the Euler equation,

$$\frac{u'(c^g)}{\beta u'(\omega - c^g)} = \frac{q+d}{q} = -\frac{\omega - c^g - \omega^b + Ad}{c^g - \omega^g + dA}$$

rearranging terms,

$$F^L(c^g) = u'(c^g)(c^g - \omega^g + dA) + \beta u'(\omega - c^g)(\omega - c^g - \omega^b + Ad)$$

If d is sufficiently large, $F^L(\frac{\omega}{2}) > 0$ and the symmetric first-best is the unique steady state. When, $A = 0$, we obtain the same solution as in the previous section.

Proposition 3: There exists a unique level of debt $\widehat{d}$ so that $F^L(c^g) = 0$, where c^g also solves $F^D(c^g) = 0$.

We can write the equilibrium prices as follow, let $Z = u'(\omega - c^g)/u'(c^g) > 1$. Then, the implied equilibrium price in a symmetric equilibrium is

$$p = \frac{\widetilde{\beta}}{1 - \widetilde{\beta}} d$$

where $\widetilde{\beta} = Z\beta$. The implied equilibrium prices depend on on Z. If the borrowing constraint binds $Z > 1$, and makes the effective discount rate larger $\widetilde{\beta} > \beta$. When the borrowing constraint does not bind $Z = 1$, so we have the complete markets solution. In the next section, we explore an economy where shock are not transitory.

7.3 Stochastic Liquidity Constrained Economy

We assume that shock can persist for several periods. In particular assume a symmetric shock

$$\Pi_{w'/w} = \begin{bmatrix} \pi_{gg} & \pi_{bg} \\ \pi_{gb} & \pi_{bb} \end{bmatrix} = \begin{bmatrix} 1-\pi & \pi \\ \pi & 1-\pi \end{bmatrix},$$

We begin by defining a competitive equilibrium in this class of economies.

Definition: *A competitive equilibrium in the stochastic economy is an contingent consumption allocation $\{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ a portfolio decision $\{\{s_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^{2}$, and state contingent prices $\{p_t\}_{t=0}^{\infty}$, st.*

- *Consumers solve*

$$\max(1-\beta)E_0\sum_{t=0}^{\infty}\beta^t u(c_t^i)$$

$$\begin{aligned} s.t. \qquad c_t^i + q_t s_{t+1}^i &\leq w_t^i + (q_t+d)s_t^i \qquad \forall t \\ s_t^i &\geq 0, \qquad s_0^i \text{ given} \end{aligned}$$

- *Markets clear*

$$\begin{aligned} c_t^1 + c_t^2 = \omega^g + \omega^b + d &= \omega \qquad \forall t \\ s_t^1 + s_t^2 &\leq 1 \qquad \forall t \end{aligned}$$

Again we focus all the attention the symmetric steady state of both economies. We want to compute the decision rules for both shocks.

$$c_t^i = \begin{cases} c^g \text{ if } w_t^i = \omega^g \\ c^b \text{ if } w_t^i = \omega^b \end{cases}$$

When the borrowing costraint does not bind we have a symmetric steady state. in the absence of aggregate uncertainty, the equilibrium price is determined by the Euler equation of both consumers. Formally,

$$\frac{u'(c^g)}{(1-\pi)u'(c^g)+\pi u'(c^b)} = \frac{p+d}{p} = \frac{u'(c^b)}{(1-\pi)u'(c^b)+\pi u'(c^g)},$$

where $c^g = c^b = \omega/2$, so $u'(c^g) = u'(c^b)$. Therefore,

$$p = \frac{\beta}{1-\beta}d.$$

However, when the borrowing constraint binds, we have

$$\frac{u'(c^g)}{(1-\pi)u'(c^g)+\pi u'(c^b)} = \beta\frac{(p+d)}{p} \Longrightarrow s^g = 1,$$

$$\frac{u'(c^b)}{(1-\pi)u'(c^b)+\pi u'(c^g)} > \beta\frac{(p+d)}{p} \Longrightarrow s^b = 0,$$

Then, substituting the consumer budget constraint for the agent with the good shock

$$c^g + q = \omega^g \Rightarrow q = \omega^g - c^g,$$

and the agent with the bad shock

$$c^b = \omega^b + (q+d) \Rightarrow q + d = c^b - \omega^b$$

we obtain,

$$\frac{u'(c^g)}{(1-\pi)u'(c^g)+\pi u'(c^b)} = \frac{\beta(c^b-\omega^b)}{\omega^g - c^g}$$

Rearranging terms we have

$$F(c^g, c^b) = u'(c^g)(\omega^g - c^g) - \beta(c^b - \omega^b)((1-\pi)u'(c^g) + \pi u'(c^b))$$

if we substitute the aggregate resource constraint $c^b = \omega - c^g$ we have

$$F(c^g) = u'(c^g)(\omega^g - c^g) - \beta(\omega - c^g - \omega^b)((1-\pi)u'(c^g) + \pi u'(\omega - c^g))$$

we obtain the solution without uncertainty as a special case where $\pi = 0$. to compute the equilibrium, we only need to solve this system with one equation and one unknown. This model implies an stochastic discount factor different than one. Formally, the pricing agent has

$$m^g = \beta\frac{(1-\pi)u'(c^g)+\pi u'(c^b)}{u'(c^g)} = (1-\pi)\beta + \pi\beta\frac{u'(c^b)}{u'(c^g)},$$

$$m^b = \beta\frac{(1-\pi)u'(c^b)+\pi u'(c^g)}{u'(c^b)} = (1-\pi)\beta + \pi\beta\frac{u'(c^g)}{u'(c^b)},$$

then we have that $m^g > m^b$ because $u'(c^b) > u'(c^g)$. In incomplete markets, the pricing agent is the individual with the highest stochastic discount factor. We can rewrite the pricing equations as

$$p = \max\{\frac{m^g}{1-m^g}, \frac{m^b}{1-m^b}\} \cdot d$$

Notice that equilibrium prices depend on the consumption allocations for both agents, and this depend on the source of uncertainty.

The equilibrium allocations for this economy when the borrowing constraint binds are given by $\{\widehat{c}^g, \widehat{c}^b\}$, the optimal portfolio allocations $\widehat{s}^g = 1$, $\widehat{s}^b = 0$, and the equilibrium price. When the borrowing constraint does not bind, $\widehat{c}^g = \widehat{c}^b = \omega/2$, and portfolio satisfies an interior solution.

just like in the previous section, we could relax the borrowing constraint, and assume $s \geq -A$. Next models, considers endogenous borrowing constraints.

7.4 Equilibrium Prices in a Debt Constrained Economy

Next, we explore an economy where the borrowing constraints are endogenously determined. At any point in time, households have an incentive to renege on their claims and walk away from the credit market. The punishment from defaulting in credit market is that a household is excluded from future intertemporal trade. Formally, the individual rationality constraint implies

$$(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}u(c_\tau^i) \geq (1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}u(\omega_\tau^i) \qquad \forall t,$$

The value of continuing participating in the market is no less that the value of dropping out. The credit agency will never lend so much to the consumers so they will choose bankruptcy. Next, we define the notion of market equilibrium. We have assumed that the individual rationality constraint is directly imposed into the consumer budget constraint.

Definition: *A competitive equilibrium in this economy is an allocation* $\{c_t^1, c_t^2\}_{t=0}^{\infty}$, *and prices* $\{p_t\}_{t=0}^{\infty}$, *such that.*

- *Consumers i solves*

$$\max(1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t^i)$$

$$s.t. \qquad \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t(w_t^i + s_0^i d) \qquad \forall t$$

$$(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}u(c_\tau^i) \geq (1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}u(w_\tau^i)$$

- *Markets clear*

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \qquad \forall t$$

$$s_t^1 + s_t^2 \leq 1 \qquad \forall t$$

Let λ and γ_t be the Lagrange multipliers of the Arrow-Debreu resource constraint, and the participation constraint respectively. Then, the first-order conditions of the consumer problem are given by

$$(1-\beta)\beta^t u'(c_t^i) - \lambda p_t + \gamma_t(1-\beta)u'(c_t^i) \leq 0,$$

We can consider two solutions of the consumer problem.

7.4.1 Debt constraint does not bind ($\gamma_t = 0$)

In this case, the friction is not operative and consumers can obtain an equilibrium allocation with perfect smoothing, or risk sharing in the case of uncertainty. We have the standard Euler equation

$$\frac{u'(c_t^i)}{\beta u'(c_{t+1}^i)} = \frac{p_t}{p_{t+1}},$$

In the symmetric equilibrium ($c^b = c^g = c^*$), under

$$\frac{u'(c^g)}{\beta u'(c^b)} = \frac{u'(c^b)}{\beta u'(c^g)} = \frac{u'(c^*)}{\beta u'(c^*)} = \frac{p_t}{p_{t+1}},$$

Hence, the equilibrium prices are given by

$$p_{t+1} = \beta p_t,$$

or

$$p_t = \beta^t p_0,$$

where $p_0 = 1$. In this case, no one has an incentive to default in their payments, even though there is no commitment on the financial market.

7.4.2 Debt constraint does bind ($\gamma_t > 0$)

In some cases it might be impossible to reach a symmetric steady state without violating the individual rationality constraint. The consumer with a good productivity shock, ω^g, after having received several bad income shock has to repay to the individual with a bad productivity shock. In this case, the individual rationality constraint is violated, because the consumer that receives the good shock prefers to declare default rather than honor its debt. Hence, the individual rationality constraint must bind exactly.

In a symmetric equilibrium we can rewrite the participation constraint

$$(1-\beta)\sum_{j=0}^{\infty}\beta^{j-1}u(c_{t+j}) \geq (1-\beta)\sum_{j=0}^{\infty}\beta^{j-1}u(w_{t+j})$$

as follows

$$\sum_{j=0}^{\infty}\beta^{2j}u(c^g) + \sum_{j=0}^{\infty}\beta^{2j+1}u(c^b) \geq \sum_{j=0}^{\infty}\beta^{2j}u(w^g) + \sum_{j=0}^{\infty}\beta^{2j+1}u(w^b)$$

$$(1-\beta)[\frac{u(c^g)}{1-\beta} + \frac{\beta u(c^b)}{1-\beta}] \geq (1-\beta)[\frac{u(w^g)}{1-\beta} + \frac{\beta u(w^b)}{1-\beta}]$$

so we obtain the participation constraint for the agent that receives the good shock in the existing period,

$$u(c^g) + \beta u(c^b) \geq u(w^g) + \beta u(w^b),$$

and the participation constraint for the agent that receives the bad income shock

$$u(c^b) + \beta u(c^g) \geq u(w^b) + \beta u(w^g),$$

When the participation constraint binds, the consumption distribution is determined by the participation and the aggregate resource constraint. Formally,

$$F^D(c^g) = u(c^g) - u(\omega^g) + \beta\left[u(\omega - c^g) - u(\omega^b)\right]$$

The equilibrium consumption depends on the income spread, $\triangle\omega = \omega^g - \omega^b$, the individual discount rate, β, and the return of the tree d. The equilibrium with imperfect risk sharing implies $c^g > c^b$.

We compute the equilibrium asset prices using the Euler equation for the consumer without a binding participation constraint. In this case the consumer with the low income shock

$$\frac{p+d}{p} = \frac{u'(c^b)}{\beta u'(c^g)} = \frac{1}{\beta A}$$

where $1/A = u'(c^b)/u'(c^g)$, given that $c^g > c^b$ it must be the case that $u'(c^g) < u'(c^b)$, hence, $A < 1$. The implied equilibrium prices depend on A

$$p = \frac{\beta A}{1-\beta A}d,$$

with complete markets $A = 1$, so we would obtain the same prices. Next, we want to show that the implied equilibrium return is lower that the inverse of the discount rate. If we consider the Euler equation of the individual with a binding participation constraint we have

$$\frac{u'(c^g)}{\beta u'(c^b)} > \frac{p+d}{p} = 1+r$$

or

$$1+r < \frac{1}{\beta}$$

Proposition 4: A symmetric steady state on the debt constraint economy is characterized by

- If the participation constraint binds, $F^D(c^g) = 0$, $c^g > c^b$
- If the participation constraint does not bind, $F^D(c^g) \geq 0$, $c^g = c^b = w/2$.

In the debt constrained economy changes in the discount rate increase the penalty from being excluded from intertemporal trade. However, full efficient allocations can be achieved if individuals are sufficiently patient. Changes in the return of the tree, increase the penalty of loosing your collateral if you default. Finally, the implied equilibrium interest rate is lower than with complete markets or perfect risk sharing.

7.4.3 Pareto Efficiency

We are interested in the welfare properties of the allocations in the debt constrained economy. In a symmetric steady state, the set of Pareto efficient allocations is characterized by solving

$$\max \lambda u(c^g) + (1-\lambda)u(c^b)$$

$$\begin{aligned} s.t. \quad & c^g + c^b = \omega = \omega^g + \omega^b + d, \\ & u(c^g) + \beta u(c^b) \geq u(w^g) + \beta u(w^b), \\ & u(c^b) + \beta u(c^g) \geq u(w^b) + \beta u(w^g), \end{aligned}$$

Notice that we have included the participation constraints as part of the feasible set of the social planner problem. Given that agents trade is voluntarily, they should obtain gains from trade. If we substitute the aggregate resource constraint and rewrite the problem as

$$\max \lambda u(c^g) + (1-\lambda)u(\omega - c^g)$$

$$\begin{aligned} s.t. \quad & u(c^g) + \beta u(\omega - c^g) \geq u(w^g) + \beta u(w^b), \\ & u(\omega - c^g) + \beta u(c^g) \geq u(w^b) + \beta u(w^g), \end{aligned}$$

Let γ_t^1 and γ_t^2 be the Lagrange multiplier of the participation constraint of both agents. The first-order conditions of the social planner problem are given by

$$\lambda u'(c^g) - (1-\lambda)u'(\omega - c^g) + \gamma_t^1[u'(c^g) - \beta u'(\omega - c^g)] - \gamma_t^2[u'(\omega - c^g) - \beta u'(c^g)] = 0$$

We can rearrange terms

$$(\lambda + \gamma_t^1 + \beta\gamma_t^2)u'(c^g) = (1 - \lambda + \gamma_t^2 + \gamma_t^1\beta)u'(\omega - c^g),$$

Notice that in this problem the planning weights are endogenous to the problem. When the participation constraint binds for one agent. The social planner needs to assign him more consumption today to keep him in the trading arrangement. When $\gamma_t^1 = \gamma_t^2 = 0$, the optimal allocation implies perfect intertemporal smoothing, or perfect risk sharing with symmetric weights ($\lambda = 1/2$). Formally,

$$u'(c^g) = u'(\omega - c^g) \Rightarrow c^g = c^b = \frac{\omega}{2},$$

However, when the participation constraint binds for the agent that had the good shock today $\gamma^1 > 0$, then the constrained efficient allocation implies imperfect smoothing, or risk sharing. Formally,

$$(\lambda + \gamma^1)u'(c^g) = (1 - \lambda + \gamma^1\beta)u'(\omega - c^g)$$

that is

$$\frac{u'(c^g)}{u'(\omega - c^g)} = \frac{\lambda + \gamma^1\beta}{1 - \lambda + \gamma^1} < 1$$

when we consider symmetric weights

$$u'(c^g) < u'(\omega - c^g) \;\Rightarrow c^g > c^b,$$

Finally, we explore the welfare properties of Pareto efficient allocations. In particular, we prove the fist-welfare theorem.

Proposition: *An equilibrium allocation in the debt constraint economy* $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$ *is Pareto efficient.*

Proof: Suppose the contrary, then there exists a Pareto superior allocation $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$, that satisfies the participation constraints. At the equilibrium prices $\{p_t\}_{t=0}^{\infty}$, this allocation has to cost strictly more than the endowment for the individual that is better of (suppose agent 1), otherwise this agent is not maximizing his utility. That is,

$$\sum_{t=0}^{\infty} p_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} p_t (w_t^1 + \theta_0^1 d)$$

Using the same argument for the other consumer (agent 2), this allocation needs to be at least as expensive as the endowment. Formally,

$$\sum_{t=0}^{\infty} p_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} p_t (w_t^2 + \theta_0^2 d)$$

If we add up both constraints we find,

$$\sum_{t=0}^{\infty} p_t [\tilde{c}_t^1 + \tilde{c}_t^2] > \sum_{t=0}^{\infty} p_t \underbrace{[w_t^1 + w_t^2 + (\theta_0^1 + \theta_0^2) d]}$$

using market clearing condition in the asset market $\theta_0^1 + \theta_0^2 = 1$, and substituting each period resource constraint $\omega = w_t^1 + w_t^2 + d$.

$$\sum_{t=0}^{\infty} p_t [\tilde{c}_t^1 + \tilde{c}_t^2] > \sum_{t=0}^{\infty} p_t \omega$$

This alternative allocation $\{\tilde{c}_t^1, \tilde{c}_t^2\}_{t=0}^{\infty}$ costs more than the endowment. Then, the allocation cannot be feasible, which contradicts the assumption of Pareto superior allocation.

Now we turn the attention to economies with uncertainty, as in the previous sections. In this environment, the value associated to walk away is given by

$$v^{AUT} = E_{t-1} \sum_{t=0}^{\infty} \beta^t u(w_t)$$

The financial contracts that satisfy the endogenous debt constraint are given by

$$u(c_t) + \beta E_{j-1} \sum_{j=1}^{\infty} \beta^{j-1} u(c_{t+j}) \geq u(w_t) + \beta v^{AUT}$$

using the previous notation, or

$$(1-\beta) E_{j-1} \sum_{j=0}^{\infty} \beta^{j-1} u(c_{t+j}) \geq (1-\beta) E_{j-1} \sum_{j=0}^{\infty} \beta^{j-1} u(w_{t+j}) \qquad \forall t$$

7.5 Stochastic Debt Constrained Economy

We assume that shock can persist for several periods. In particular assume a symmetric shock

$$\Pi_{w'/w} = \begin{bmatrix} \pi_{gg} & \pi_{bg} \\ \pi_{gb} & \pi_{bb} \end{bmatrix} = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix},$$

We begin by defining a competitive equilibrium in this class of economies.

Definition: *A competitive equilibrium in the stochastic economy is an contingent consumption allocation $\{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^2$ and state contingent prices $\{p_t\}_{t=0}^{\infty}$, st.*

- *Consumers solve*

$$\max (1-\beta) E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

$$s.t. \qquad E_0 \sum_{t=0}^{\infty} p_t c_t^i \leq E_0 \sum_{t=0}^{\infty} p_t (w_t^i + \theta_0^i d) \qquad \forall t$$

$$(1-\beta) E_t \sum_{j=0}^{\infty} \beta^{j-1} u(c_{t+j}) \geq (1-\beta) E_t \sum_{j=0}^{\infty} \beta^{j-1} u(w_{t+j}) \qquad \forall t \geq 0$$

- *Markets clear*

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \qquad \forall t$$

As in the previous case, we want to focus the attention on the steady state of both economies. We want to compute the decision rules for both shocks. Now its agent is going to face the good shock with a certain probability. For simplicity we drop the time index and all the notation is contingent the shock. In a symmetric steady state

$$c^i(s) = \begin{cases} c^g \text{ if } w^i(s) = \omega^g \\ c^b \text{ if } w^i(s) = \omega^b \end{cases}$$

The stochastic steady state is like the deterministic case. We lower c^g from the individual with the good productivity shock until,either the symmetric first-best $c^g = \omega/2$ is achieved or the participation constraint binds. For the stochastic case, we can also compute the expected utility associated to the symmetric steady state, where π denotes the probability of continue in the same state, and $1-\pi$ denotes the probability of reversal.

$$E_t \sum_{j=0}^{\infty} \beta^{j-1} u(c_{t+j}) = u(c^g) + \beta[\pi u(c^g) + (1-\pi) u(c^b)] + ...$$
$$\beta^2[\pi^2 u(c^g) + \pi(1-\pi) u(c^g) + \pi(1-\pi) u(c^g) + (1-\pi)^2 u(c^b)] + ...$$

rearranging terms

$$u(c^g) + \beta[\pi u(c^g) + (1-\pi) u(c^b)] + \beta^2[\pi u(c^g) + (1-\pi) u(c^b)] + ...$$

that is

$$u(c^g) + \sum_{t=1}^{\infty} \beta^t \left[\pi u(c^g) + (1-\pi) u(c^b)\right] = u(c^g) + \frac{\left[\pi u(c^g) + (1-\pi) u(c^b)\right]}{1-\beta}.$$

We have a similar expression with respect to income shocks. Combing all terms we have

$$(1-\beta)u(c^g)+\beta\left[\pi u(c^g)+(1-\pi)u(c^b)\right] \geq (1-\beta)u(w^g)+\beta\left[\pi u(w^g)+(1-\pi)u(w^b)\right]$$

and as before define $F(c^g)$ as

$$F(c^g) = (1-\beta(1-\pi))[u(c^g)-u(\omega^g)] + \beta\pi[u(\omega-c^g)-u(\omega^b)]$$

Proposition: *A symmetric stochastic steady state c^g on the debt constraint economy is characterized by*

- *If the participation constraint binds, $F^D(c^g)=0$, $c^g > c^b$*
- *If the participation constraint does not bind, $F^D(c^g) \geq 0$, $c^g = c^b = w/2$.*

When $\pi = 1$ the function F^D is concave and satisfies $F^D(\omega^g) > 0$, so the symmetric steady state existed and is unique. For $\pi \in (0,1)$ this is still true and we reach the same conclusions.

Now we want to explore the effect on the equilibrium allocations depends on the parameter $1-\pi$ that measures the persistence of the shock. From the implicit function theorem we can compute $\partial c^g/\partial(1-\pi)$. A useful way is to rewrite the function F^D as a function of π.

$$F^D(c^g) = (1-\beta)[u(c^g)-u(w^g)] + \beta\pi[u(\omega-c^g)-u(\omega^b)+u(c^g)-u(\omega^b)]$$

when the participation constraint binds, $F^D(c^g)=0$. The first term is always negative $(u(c^g)-u(w^g)<0)$, and the second term is always positive, $u(\omega-c^g)-u(\omega^b)>0$ and $u(c^g)-u(\omega^b)>0$. Since $\partial c^g/\partial\pi$ is proportional to the second term,

$$\frac{\partial c^g}{\partial \pi} = \beta[u(\omega-c^g)-u(\omega^b)+u(c^g)-u(\omega^b)] > 0$$

to show that $\partial c^g/\partial(1-\pi) > 0$, we have to redefine the function F^D.

$$F(c^g) = (1-\beta(1-\pi))[u(c^g)-u(\omega^g)] - \beta\pi[u(\omega^b)-u(\omega-c^g)] = ...$$

$$F(c^g) = (1-\beta(1-\pi))[u(c^g)-u(\omega^g)] - \beta\pi[u(\omega^b)-u(\omega-c^g)] - \beta[u(\omega^b)-u(\omega-c^g)] + \beta[u(\omega^b)-u(\omega-c$$

rearranging terms

$$F(c^g) = (1-\beta(1-\pi))[u(c^g)-u(\omega^g)] - \beta(1-\pi)[u(\omega^b)-u(\omega-c^g)] + \beta[u(\omega^b)-u(\omega-c^g)]$$

$$F(c^g) = [u(c^g)-u(\omega^g)] + \beta[u(\omega^b)-u(\omega-c^g)] - \beta(1-\pi)\left[u(c^g)-u(\omega^g)+u(\omega^b)-u(\omega-c^g)\right]$$

or

$$F(c^g) = [u(c^g)-u(\omega^g)] + \beta[u(\omega^b)-u(\omega-c^g)] + \beta(1-\pi)\left[u(\omega^g)-u(c^g)+u(\omega-c^g)-u(\omega^b)\right]$$

where

$$\frac{\partial c^g}{\partial(1-\pi)} = \beta\left[\underbrace{u(\omega^g)-u(c^g)}_{>0} + \underbrace{u(\omega-c^g)-u(\omega^b)}_{>0}\right] > 0$$

This result implies that a more persistent shock results in greater consumption by the individual with the high productivity shock, or equivalently less trade between two consumers. So in this economy, when consumption is stochastic the amount of consumption smoothing is reduced.

Although this decentralization works without problems, it conflicts with the spirit that at every time and contingency, households should be able to walk away from the contract. In this environment, all decisions are made at $t=0$, so households cannot choose to renege on the time 0 contingent contracts because they confront no choices from period 0 onwards. This critique has been addressed by Alvarez and Jermann (2000), that solve the decentralization in terms of sequential trading.

7.6 Financial Intermediation without Commitment

- Discrete time periods $t = 0, 1, ...$
- Large number of ex-ante identical households
- Single consumption good c_t.
- Infinite vector of consumption $(c_0, c_1, ...) \in l_\infty^{++}$.
- Preferences

$$U(c_0, c_1, ...) = E\sum_{t=0}^{\infty}\beta^t u(c_t)$$

- $u' > 0$, $u'' < 0$, Inada conditions and $\beta \in (0,1)$.
- Each household receives an stochastic endowment $\{y_t\}_{t=0,}^{\infty}$ where $y_t \tilde{} i.i.d.$
- Denote $\pi(s) = \Pr ob(y_t = \overline{y}_s)$, with finite support $s \in \{1, 2, ..., S\}$ and $\overline{y}_{s+1} > \overline{y}_s$
- History of endowments is given by $h^t = (y_t, y_{t-1}, ..., y_0)$
- Moneylender or financial intermediary has access to an storage technology and can borrow or lend at a risk free rate $R = \beta^{-1} > 1$
- Consumers can only deal with the financial intermediary, they cannot trade among themselves.
- The moneylender designs a contract,

$$c_t = f_t(h^t) \qquad t \geq 0$$

that specifies a sequence of functions that assign history dependent consumption. Therefore, consumers give the endowment to the moneylender and then they receive some consumption in exchange. The purpose of the contract is to smooth consumption over time. The revenues and the utility associated to a particular contract are given by

$$P = E\sum_{t=0}^{\infty}\beta^t[y_t - c_t] = E\sum_{t=0}^{\infty}\frac{1}{R^t}[y_t - c_t]$$
$$v = E\sum_{t=0}^{\infty}\beta^t u(c_t) = E\sum_{t=0}^{\infty}\beta^t u[f_t(h^t)]$$

where P denotes the associated profits and v denotes the utility associated to the moneylender contract.

7.6.1 Risk Sharing with Full Commitment

In this section we study risk sharing contracts with two-sided commitment, that means both agents are obliged to satisfy the contract after it has been signed. Alternatively, we can think of an infinite penalty for breaking the

relationship at some point in the event tree. The optimal risk sharing contract solves

$$\max_{\{c_t\}} P = E\sum_{t=0}^{\infty}\beta^t[y_t - c_t]$$
$$s.t. \qquad E\sum_{t=0}^{\infty}\beta^t u(c_t) = v$$
$$c_t \geq 0$$

or developing the expectation operator

$$\max_{\{c_t(s^t)\}} P = \sum_{t=0}^{\infty}\sum_{s=1}^{S}\pi(s)\beta^t[y(s_t) - c_t(s^t)]$$
$$s.t. \qquad \sum_{t=0}^{\infty}\sum_{s=1}^{S}\pi(s)\beta^t u[c_t(s^t)] = v$$
$$c_t(s^t) \geq 0$$

The constraint set is convex and the objective function is concave. Hence, the optimization problems is well-defined, so we can characterize the optimal contract using the first-order conditions.

$$-\pi(s)\beta^t + \lambda\pi(s)\beta^t u'[c_t(s^t)] = 0$$
$$-\pi(\widetilde{s})\beta^t + \lambda\pi(\widetilde{s})\beta^t u'[c_t(\widetilde{s}^t)] = 0$$

Rearranging terms

$$1 = \frac{u'[c_t(s^t)]}{u'[c_t(\widetilde{s}^t)]}$$

this expression equates the marginal rate of transformation of the moneylender to the marginal rate of substitution of the consumer. In an interior solution the promise-keeping constraint will be binding. This arrangement implies that the marginal utility of the consumer is constant across states, $u'[c_t(s^t)] = u'[c_t(\widetilde{s}^t)]$, which implies that consumption should be constant too, $c_t(s^t)] = c_t(\widetilde{s}^t)$. Therefore, the moneylender perfectly insures the consumer across time and states of the nature.

7.6.2 Risk Sharing with One-sided Commitment

Now we assume that the financial intermediary is committed to honor the promises but the consumers can walk away from the contract at any time, this is called one-sided commitment contracts. Therefore, the contract the planner (moneylender) offers must be "self-enforcing" in the face of lack of commitment.

$$v^{AUT} = E\sum_{t=0}^{\infty}\beta^t u(y_t)$$

denote the expected utility associated to receive the endowment. Then, at any point in time consumers can receive

$$u(y_t) + \beta v^{AUT}$$

If the financial intermediary wants to induce the households to trade it has to offer him a better contract. Formally,

$$u(c_t) + \beta E\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j}) \geq u(y_t) + \beta v^{AUT}$$

or using the definition of a contract, $c_t = f_t(h^t)$

$$u[f_t(h^t)] + \beta E\sum_{j=1}^{\infty}\beta^{j-1}u[f_{t+j}(h^{t+j})] \geq u(y_t) + \beta v^{AUT}$$

This is the participation constraint and make a contract sustainable, in the sense that the individual does not have an incentive to walk away from the contract. The problem with this constraint is that depends on the history h^t and that grows rapidly overtime t. Now the optimal contract has to solve

$$\max_{\{c_t\}} P = E\sum_{t=0}^{\infty}\beta^t[y_t - c_t]$$

$$s.t. \qquad E\sum_{t=0}^{\infty}\beta^t u(c_t) = v$$

$$u(c_t) + \beta E\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j}) \geq u(y_t) + \beta v^{AUT}$$

$$c_t \geq 0$$

7.6.3 Promised Utility Formulation

To make this problem simple we will use a recursive formulation of history dependent contract that implies enlarge the state space by redefining a new variable v_t, that represents the promised discounted future value or utility. Define the optimal contract (i.e. the policy functions) associate to this problem as

$$c_t = g(y_t, v_t)$$
$$v_{t+1} = l(y_t, v_t)$$

where the optimal contract depend on the current endowment and the history of shock summarized by v_t. Iterating on v_t we can back up the history of shocks,

$$v_1 = l(y_0, v_0)$$
$$v_2 = l(y_1, v_1) = l(y_1, y_0, v_0)$$
$$v_3 = l(y_2, v_2) = l(y_2, y_1, y_0, v_0)$$
$$......$$
$$v_t = l(y_{t-1}, v_{t-1}) = l(y_{t-1}, y_{t-2}, ..., y_1, y_0, v_0)$$

The planner gives to the household a particular utility level v by delivering state contingent consumption assigned by the contract and promises some utility tomorrow, defined by $v' = w_s$. The state variable in the optimal contract problem is the promised level of utility. The money lender problem has to be a strictly decreasing function of v. The higher this value the smaller the profits that the planner will receive by trading. Using recursive notation we can redefine the optimal contract problem,

$$P(v) = \max_{\{c_t\}} E\left[[y_t - c_t] + \beta P(w_s)\right]$$

$$s.t. \qquad E[u(c_t) + \beta w_s] = v$$

$$u(c_t) + \beta w_s \geq u(\overline{y}_t) + \beta v^{AUT} \qquad \forall s$$

$$c_t \geq 0$$

or

$$P(v) = \max_{\{c(s), w_s\}} \sum_{s=1}^{S}\pi_s\left[[y_s - c_s] + \beta P(w_s)\right]$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s[u(c_s) + \beta w_s] = v$$
$$u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v^{AUT} \qquad \forall s$$
$$c_s \in [c_{\min}, c_{\max}]$$
$$w_s \in [v^{AUT}, \overline{v}]$$

Again, the constraint set is convex and the return function is concave, therefore the value function $P(v)$ is concave. The Lagrangian of the recursive contract can be written as follows.

$$L = \sum_{s=1}^{S} \pi_s [[y_s - c_s] + \beta P(w_s)] + \mu \left[\sum_{s=1}^{S} \pi_s [u(c_s) + \beta w_s] - v \right] +$$
$$\sum_{s=1}^{S} \lambda_s \left[u(c_s) + \beta w_s - [u(\overline{y}_s) + \beta v^{AUT}] \right]$$

the first order conditions with respect to $\{c_s, w_s\}$ are given by

$$-\pi_s + \mu \pi_s u'(c_s) + \lambda_s u'(c_s) = 0$$
$$\pi_s \beta P'(w_s) + \mu \pi_s \beta + \lambda_s \beta w_s = 0$$

and using the envelope theorem we can compute the change in the profit function associated to a change in the period promised value v,

$$\pi_s \beta P'(v) + \mu \pi_s \beta = 0$$

Rearranging terms

$$(\mu \pi_s + \lambda_s) u'(c_s) = \pi_s$$
$$(\mu \pi_s + \lambda_s) = -\pi_s P'(w_s)$$
$$P'(v) = -\mu$$

Given that the profit function is decreasing in v. Then $P'(v) < 0$, which means that the Lagrange multiplier of the promise-keeping constraint has to be positive $\mu > 0$. Given that $(\mu \pi_s + \lambda_s)$ has to be positive and $\pi_s \geq 0$, then it must be the case that $\lambda_s > 0$. Combing both expressions we have

$$u'(c_s) = -\frac{1}{P'(w_s)}$$

This expression equates the marginal rates of substitution between contingent consumption today and promised utility to the marginal rate of transformation for the planner of tomorrows utility. This equation has a positive slope in c_s and w_s. It is important to not, that $P' < 0$ is decreasing in w_s, but the inverse must be increasing, and the negative in front of it changes the sign of the expression. The dynamic equation is given by a trade of between promised value today and tomorrow,

$$P'(w_s) = P'(v) - \frac{\lambda_s}{\pi_s}$$

What will happens to the promised value utility depends on the Lagrange multiplier of the participation constraint.

- **Participation constraint binds** ($\lambda_s > 0$)

  If the participation constraint binds, this is because the consumer has received a good income shock and has to return an important part of the endowment to the financial intermediary in exchange. It is important to remark that previous to this event, the consumer had received bad income shocks. The one-side commitment problem introduces incentives to walk away from the contract. To prevent that the planner has to promise higher expected utility in the future. That should be more clear from the above equations,

$$P'(w_s) < P'(v) \Rightarrow w_s > v$$

  this is true because of the concavity of the function P, that implies $c_s \leq \overline{y}_s$. The planner induces the household to consume less by promising more utility tomorrow, that is w_s. The optimal level of consumption c_s and w_s can be determined

$$u'(c_s) = -P'(w_s)^{-1}$$
$$u(c_s) + \beta w_s = u(\overline{y}_s) + \beta v^{AUT}$$

  These equations are independent of v. Part of the optimal contract implies the existence of amnesia. After receiving a good shock the planner changes the promise utility from that period onwards, so the new consumption will be a function of w_s not a function of v. The solution of the optimal contract is given by

$$c_s = g_1(\overline{y}_s)$$
$$w_s = l_1(\overline{y}_s)$$

the good shock induces a higher continuation value, therefore from this point onwards history does not matter and the new continuation value defines future expected utility.

- **Participation constraint does not binds** ($\lambda_s = 0$)

  If the participation constraint does not binds, this is because the consumer has received a bad income shock. In this particular case, the consumer does not have any incentive to walk away, because the contract is going to provide consumption insurance. Hence, the planner does not need to provide incentives, because for this particular shock there is no treat to break the contract, it is not on the individuals best interest. Formally,

  $$P'(w_s) = P'(v) \Rightarrow w_s = v$$

  In this case, contingent consumption is determined using

  $$u'(c_s) = -P'(w_s)^{-1} = -P'(v)^{-1}$$

  the optimal level of consumption depends on the promised value $w_s = v$ not on a particular realization of the shock $\overline{y}_s$. The solution of the optimal contract is then given by

  $$c_s = g_2(v)$$
  $$w_s = v$$

  and

  $$u'[g_2(v_s)] = -P'(v)^{-1}$$

The "*optimal contract*" implied by

$$c = \max\{g_1(\overline{y}_s), g_2(v)\}$$
$$w_s = \max\{l_1(\overline{y}_s), v\}$$

For the interval of promised utilities $v \in (v^{AUT}, \overline{v})$ there exists a cutoff point in terms of endowment shock, $\overline{y}(v)$ such that:

- If $y \leq \overline{y}(v)$, the planner offer the contract $c = g_2(v)$ and leaves the promised utility unaltered, $w_s = v$. Thus, the planner is insuring in the states with low income shocks.

- If $y \geq \overline{y}(v)$, the participation constrain is binding, so the planner induces the consumer to surrender part of its endowment in exchange of a higher promised utility, $w_s > v$.

It is important to mention that promise utility values never decrease, stay constant if $y \leq \overline{y}(v)$ or increase if $y \geq \overline{y}(v)$ where the participation constraint is threaten to be violated. This is also called the Rachet effect, and is implied by consumption smoothing. Consumption is constant in periods where the participation constraint is not binding, because v does not change and increases in periods were it threatens to bind.

The planner has to ways to give incentives, increase present consumption and promised utility. The concave scheme on the utility function implies that the planner will have to use both if the participation constraint binds. Promising more utility in the future is not enough to prevent consumers from not walking away. Thus, the household with the high endowment, $\overline{y}_S$ is permanently awarded with the highest consumption level associated with $\overline{v}$, that is $c = g_2(\overline{v})$,

$$u(g_2(\overline{v})) + \beta\overline{v} = u(\overline{y}_S) + \beta v^{AUT}$$

where $c \leq \overline{y}_S$ but $\overline{v} > v^{AUT}$. On the other hand, the household with the lower endowment, $\overline{y}_1$ is expecting to receive more utility in the future because $u(\overline{y}_1) < Eu(y)$, adding in both sides the continuation value of autarchy we have

$$u(\overline{y}_1) + \beta v^{AUT} < E[u(y) + \beta v^{AUT}] = v^{AUT}$$

For this individual with the lowest shock, $y = \overline{y}_1$, the participation constraint is not binding

$$u(c) + \beta w = u(\overline{y}_1) + \beta v^{AUT} < v^{AUT}$$

The optimal contract trades off consumption against continuation value only for sufficiently high values of the realization of the shock y.

7.6.4 The Dual Approach

The *dual approach* of contracting theory can be applied when the principal or the planner is risk-neutral. Using this particular approach, the planner wants to minimize the cost of giving the right incentives to consumers, in this particular case preventing them from walking away from the optimal contract.

$$C(v) = \min_{\{c(s),w_s\}} \sum_{s=1}^{S} \pi_s \left[c_s + \beta C(w_s)\right]$$

$$s.t. \qquad \sum_{s=1}^{S} \pi_s [u(c_s) + \beta w_s] = v$$

$$u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v^{AUT} \qquad \forall s$$

$$c_s \in [c_{\min}, c_{\max}]$$
$$w_s \in [v^{AUT}, \overline{v}]$$

Let ϕ and η_s the Lagrange multipliers of the promise-keeping and participation constraint respectively. Then, the first order conditions with respect to $\{c_s, w_s\}$ are given by

$$\pi_s + \phi \pi_s u'(c_s) + \eta_s u'(c_s) = 0$$
$$\pi_s \beta C'(w_s) + \phi \pi_s \beta + \eta_s \beta = 0$$
$$-C'(v) - \phi = 0$$

Combing both expressions we have,

$$(\phi \pi_s + \eta_s) u'(c_s) = -\pi_s$$
$$(\phi \pi_s + \eta_s) = -\pi_s C'(w_s)$$
$$C'(v) = -\phi$$

Given that the marginal cost is positive, $C'(v) > 0$, then it must be the case that the Lagrange multiplier of the promise-keeping constraint is negative, $\phi < 0$. By the same argument, $(\phi \pi_s + \eta_s) < 0$, given that $\pi_s \geq 0$, it also must be the case that $\eta_s < 0$. Rearranging terms we have

$$u'(c_s) = \frac{1}{C'(w_s)}$$

$$C'(w_s) = C'(v) - \frac{\eta_s}{\pi_s}$$

What will happens to the promised value utility depends on the Lagrange multiplier of the participation constraint.

- **Participation constraint binds** ($\eta_s > 0$)

  Given that the Lagrange multiplier is $\eta_s < 0$, in the cost minimization problem, it must the case that $C'(w_s) > C'(v)$, so the convex cost function implies $w_s > v$. The planner increases the cost of keeping the agents with a binding participation constraint by increasing the promised utility w_s. From the other first-order condition we can back-out the consumption behavior and the participation constraint

  $$C'(w_s)u'(c_s) = 1$$

  $$u(c_s) + \beta w_s = u(\overline{y}_s) + \beta v^{AUT}$$

  If the marginal cost is increasing, then the marginal utility must be decreasing to keep the ratio constant, which implies that consumption is increasing c_s. As in the previous case, these equations are independent of v. The optimal contract implies the existence of amnesia.

- **Participation constraint does not binds** ($\eta_s = 0$)

  $C'(w_s) = C'(v) \Rightarrow w_s = v$. The individual does not have any incentive to leave the contractual risk sharing arrangement. Therefore, the cost for the planner has not changed, because it promises the same lifetime utility v. Consumption is determined using the first-order conditions of the optimal contract

  $$C'(v)u'(c_s) = 1.$$

  The optimal consumption depends on the promised value $w_s = v$ not on a particular realization of the shock $\overline{y}_s$.