Lucas Tree Models
Financial Economics II

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1. Consider and economy with a representative consumer with preferences described by $E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t)$ where $u(c_t) = \ln(c_t + \gamma)$ where $\gamma \geq 0$ and c_t denotes consumption of the fruit in period t. The sole source of the single good is an everlasting tree that produces d_t units of the consumption good in period t. The dividend process d_t is Markov, with $prob\{d_{t+1} \leq d' | d_t = d\} = F(d', d)$. Assume the conditional density $f(d', d)$ of F exists. There are competitive markets in the title of trees and in state-contingent claims. Let p_t be the price at t of a title to all future dividends from the tree.

   (a) Prove that the equilibrium price p_t satisfies

$$p_t = (d_t + \gamma) \sum_{j=1}^{\infty} \beta^t \, E_t \left(\frac{d_{i+j}}{d_{i+j} + \gamma} \right)$$

Consumer optimizes the following household problem.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t)$$

Budget constraint:

$$A_{t+1} = R_t(A_t + y_t - c_t)$$

where c_t, y_t, A_t, R_t indicate the consumption of an agent at time t, the agent's labor income, the amount of a single asset valued in units of consumption good, and the real gross rate of return on the asset between time t and t+1. The Euler equation gives the following condition.

$$u'(c_t) = E_t \beta R_t u'(c_{t+1})$$

The above equation does not spell out complete general equilibrium setups. Lucas's asset pricing model does use general equilibrium reasoning.

Lucas model assumptions:

The labor income is zero.
The durable good in the economy is only a set of trees.
Representative agent assumption.
The fruit is nonstorable.

Recall $c_t = d_t$ in a general equilibrium.
Letting $R_t = \frac{p_{t+1} + d_{t+1}}{p_t}$, the Euler equation will be :

$$E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(\frac{p_{t+1} + d_{t+1}}{p_t} \right) = 1$$

$$p_t = E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1})$$

Using the equilibrium condition $c_t = d_t$.

$$p_t = E_t \beta \frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1})$$

Since $u(c_t) = \ln(c_t + \gamma)$,

$$p_t = E_t \beta \frac{(d_t + \gamma)}{(d_{t+1} + \gamma)} (p_{t+1} + d_{t+1})$$

The price at time t+1 is as follows:

$$p_{t+1} = E_{t+1} \beta \frac{(d_{t+1} + \gamma)}{(d_{t+2} + \gamma)} (p_{t+2} + d_{t+2})$$

By plugging p_{t+1} back into p_t,

$$\begin{aligned} p_t &= E_t \beta \frac{(d_t + \gamma)}{(d_{t+1} + \gamma)} \left(\beta \frac{(d_{t+1} + \gamma)}{(d_{t+2} + \gamma)} (p_{t+2} + d_{t+2}) + d_{t+1} \right) \\ &= E_t \left(\beta \frac{(d_t + \gamma)}{(d_{t+1} + \gamma)} d_{t+1} + \beta^2 \frac{(d_t + \gamma)}{(d_{t+2} + \gamma)} d_{t+2} + \beta^2 \frac{(d_t + \gamma)}{(d_{t+2} + \gamma)} p_{t+2} \right) \end{aligned}$$

Recursively,

$$p_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{(d_t + \gamma)}{(d_{t+j} + \gamma)} d_{t+j} + \lim_{j \to \infty} E_t \beta^j \frac{(d_t + \gamma)}{(d_{t+j} + \gamma)} p_{t+j}$$

Since $\lim_{j\to\infty} E_t \beta^j \dfrac{(d_t+\gamma)}{(d_{t+j}+\gamma)} p_{t+j} = 0$, we obtain the final pricing formula.

$$p_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{(d_t+\gamma)}{(d_{t+j}+\gamma)} d_{t+j}$$

(b) Find a formula for the risk-free one-period interest rate R_{1t}. Prove that in the special case in which $\{d_t\}$ is independently and identically distributed, R_{1t} is given by $R_{1t}^{-1} = \beta\, k\,(d_t+\gamma)$, where k is a constant. Give a formula for k.

We now suppose that there are markets in one- and two-period perfectly safe loans, which bear gross rates of return R_{1t} and R_{2t}. At the beginning of time t, the returns R_{1t} and R_{2t} are known with certainty and are risk free from the viewpoint of the agents. That is, at time t, R_{1t}^{-1} is the price of a perfectly sure claim to one unit of consumption at time (t+1), and R_{2t}^{-1} is the price of a perfectly sure claim to one unit of consumption at time (t+2). The representative agent solves the following optimization problem:

$$\max_{c_t, L_{1t+1}, L_{2t+1}} E_0 \sum_{t=0}^{\infty} \beta^t\, u(c_t)$$

subject to the budget constraint:

$$c_t + L_{1t} + L_{2t} \leq d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2}$$

where L_{jt} is the amount lent for j periods at time t.

Using the Lagrange Multiplier method,

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t (d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2} - c_t - L_{1t} - L_{2t})\right)$$

By taking differentiations with respect to $\{c_t, L_{1t}, L_{2t}\}$.

$$\begin{aligned}
c_t &: \; E_0\beta^t(u'(c_t) - \lambda_t) = 0 \\
L_{1t} &: \; E_0\beta^t(\beta\lambda_{t+1}R_{1t} - \lambda_t) = 0 \\
L_{2t} &: \; E_0\beta^t(\beta^2\lambda_{t+2}R_{2t} - \lambda_t) = 0
\end{aligned}$$

Using the Markov property $E_0 = E_t$.

$$\begin{aligned} c_t &: \lambda_t = u'(c_t) \\ L_{1t} &: \lambda_t = E_t(\beta \lambda_{t+1} R_{1t}) \\ L_{2t} &: \lambda_t = E_t(\beta^2 \lambda_{t+2} R_{2t}) \end{aligned}$$

Combining the first-order conditions gives:

$$\begin{aligned} E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{1t}\right) &= 1 \\ E_t\left(\beta^2 \frac{u'(c_{t+2})}{u'(c_t)} R_{2t}\right) &= 1 \end{aligned}$$

Assuming the risk-free interest rates,

$$\begin{aligned} R_{1t}^{-1} &= E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}\right) \\ R_{2t}^{-1} &= E_t\left(\beta^2 \frac{u'(c_{t+2})}{u'(c_t)}\right) \end{aligned}$$

Since $u(c_t) = \ln(c_t + \gamma)$,

$$\begin{aligned} R_{1t}^{-1} &= E_t\left(\beta \frac{c_t + \gamma}{c_{t+1} + \gamma}\right) \\ R_{2t}^{-1} &= E_t\left(\beta^2 \frac{c_t + \gamma}{c_{t+2} + \gamma}\right) \end{aligned}$$

Recall $c_t = d_t$ in a general equilibrium.

$$\begin{aligned} R_{1t}^{-1} &= E_t\left(\beta \frac{d_t + \gamma}{d_{t+1} + \gamma}\right) \\ R_{2t}^{-1} &= E_t\left(\beta^2 \frac{d_t + \gamma}{d_{t+2} + \gamma}\right) \end{aligned}$$

By letting $k_{1t} = E_t\left(\frac{1}{d_{t+1} + \gamma}\right)$, the pricing formula can be expressed as:

$$R_{1t}^{-1} = \beta k_{1t}(d_t + \gamma)$$

(c) Find a formula for the risk-free two-period interest rate R_{2t}. Prove that in the special case in which $\{d_t\}$ is independently and identically distributed, R_{2t} is given by $R_{2t}^{-1} = \beta^2 \, k \, (d_t + \gamma)$, where k is the same constant you found in part **(b)**.

By letting $k_{2t} = E_t \left(\dfrac{1}{d_{t+2} + \gamma} \right)$, the pricing formula can be expressed as:

$$R_{2t}^{-1} = \beta^2 k_{2t}(d_t + \gamma)$$

Let me show that k_{1t} and k_{2t} are identical. Since d_t are identically distributed and follow the markov chain,

$$\begin{aligned} k_{2t} &= E_t \left(\frac{1}{d_{t+2} + \gamma} \right) \\ &= E_{t+1} \left(\frac{1}{d_{t+2} + \gamma} \right) \\ &= k_{1t} \end{aligned}$$

2. Consider the following version of the Lucas's tree economy. There are two kinds of trees. The first kind is ugly and gives no direct utility to consumers, but yields a stream of fruit $\{d_{1t}\}$, where d_{1t} denotes a positive random process obeying a first-order Markov process. The second tree is beautiful and yields utility on itself. This tree also yields a stream of the same kind of fruit d_{2t}, where it happens that $d_{2t} = d_{1t} = \left(\frac{1}{2}\right) d_t \; \forall \, t$, so that the physical yields of the two kinds of trees are equal. There is one of each tree for each N individuals in the economy. Trees last forever, but the fruit is not storable. Trees are the only source of fruit.

   Each of the N individuals in the economy has preferences described by

$$E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t, s_{2t}) \tag{1}$$

   where $u(c_t, s_{2t}) = \ln c_t + \gamma \ln(s_{2t})$ where $\gamma \geq 0$, c_t denotes consumption of the fruit in period t and $s2t$ is the stock of beautiful trees owned at the beginning of the period t. The owner of a tree of either kind i at the start of the period receives the fruit d_{it} produced by the tree during that period.

Let p_{it} be the price of a tree of type i (where $i = 1, 2$) during period t. Let R_{it} be the gross rate of returns of tree i during that period held from period t to $t + 1$.

(a) Write down the consumer optimization problem in sequential and recursive form.

Consumer optimization in a recursive form

The Bellman's equation is given by

$$v(d_t, s_{1t}, s_{2t}) = \max_{\{c_t, s_{1t+1}, s_{2t+1}\}} (\ln(c_t) + \gamma \ln(s_{2t}) + E_t \beta v(d_t, s_{1t+1}, s_{2t+1}))$$

where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

Consumer optimization in a sequential form

The sequential form is given by

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t (\ln(c_t) + \gamma \ln(s_{2t}))$$

where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

(b) Define a rational expectations equilibrium.

Definition The following is called the *market clear condition*

$$\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} d_t^i \tag{2}$$

$$\begin{aligned} \sum_{i=1}^{I} s_{1t}^i &= \sum_{i=1}^{I} s_{10}^i = I \\ \sum_{i=1}^{I} s_{2t}^i &= \sum_{i=1}^{I} s_{20}^i = I \end{aligned}$$

where s^i_{10} and s^i_{20} are each agent's number of trees at initial time.

Definition *A sequential household problem* is defined by each agent's utility optimization problem:

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(c_t) + \gamma \ln(s_{2t})\right)$$

where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

Definition A *rational competitive equilibrium* is an allocation, $\{\{c^i_t\}^{\infty}_{t=0}\}^I_{i=1}$, $\{\{s^i_{1t}, s^i_{2t}\}^{\infty}_{t=0}\}^I_{i=1}$, and a price system, $\{p_{1t}, p_{2t}\}^{\infty}_{t=0}$, such that the allocation solves each household problem and satisfies the market clear condition.

(c) Find the pricing functions mapping the state of the economy at t unto p_{1t} and p_{2t} (give precise formulas). [Hint: You should be able to directly derive p_{1t} from the example seen in class, then since pricing function have to be linear you can guess a pricing function $p_{2t} = kd_t$ and solve for k parameter using Euler equation of the second stock.]

I am going to use the sequential form to find a solution. Using the Lagrange Multiplier,

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \left((\ln(c_t) + \gamma \ln(s_{2t})\right) \\ &+ \lambda_t((d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t} - c_t - p_{1t}s_{1t+1} - p_{2t}s_{2t+1})) \end{aligned}$$

By taking differentiations with respect to $\{c_t, s_{1t+1}, s_{2t+1}\}$.

$$\begin{aligned} c_t &: E_0\beta^t(\frac{1}{c_t} - \lambda_t) = 0 \\ s_{1t+1} &: E_0\beta^t(\beta\lambda_{t+1}(d_{1t+1} + p_{1t+1}) - \lambda_t p_{1t}) = 0 \\ s_{2t+1} &: E_0\beta^t\left(\beta\lambda_{t+1}(d_{2t+1} + p_{2t+1}) - \lambda_t p_{2t} + \beta\frac{\gamma}{s_{2t+1}}\right) = 0 \end{aligned}$$

Using the Markov property $E_0 = E_t$.

$$\begin{aligned}
c_t &: \frac{1}{c_t} = \lambda_t \\
s_{1t+1} &: p_{1t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t}(d_{1t+1} + p_{1t+1}) \\
s_{2t+1} &: p_{2t} = E_t \beta \frac{\lambda_{t+1}}{\lambda_t}(d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma}{s_{2t+1}\lambda_t}
\end{aligned}$$

Combining the first-order conditions, we can obtain the pricing formula.

$$\begin{aligned}
p_{1t} &= E_t \beta \frac{c_t}{c_{t+1}}(d_{1t+1} + p_{1t+1}) \\
p_{2t} &= E_t \beta \frac{c_t}{c_{t+1}}(d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma c_t}{s_{2t+1}}
\end{aligned}$$

Recall $c_t = d_t$ in a general equilibrium.

$$\begin{aligned}
p_{1t} &= E_t \beta \frac{d_t}{d_{t+1}}(d_{1t+1} + p_{1t+1}) \\
p_{2t} &= E_t \beta \frac{d_t}{d_{t+1}}(d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}}
\end{aligned}$$

Let us assume the linear form of the first tree's pricing function.

$$p_{1t} = k_{1t} d_t$$

Next apply the above linear form to the Euler equation.

$$\begin{aligned}
k_{1t} d_t &= E_t \beta \frac{d_t}{d_{t+1}}(d_{1t+1} + k_{1t+1} d_{t+1}) \\
k_{1t} &= E_t \beta k_{1t+1} + E_t \beta \frac{d_{1t+1}}{d_{t+1}}
\end{aligned}$$

Recursively we obtain k_{1t}.

$$k_{1t} = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_{1t+j}}{d_{t+j}}$$

Since we are given $d_{1t} = \frac{1}{2}d_t$.

$$\begin{aligned} k_{1t} &= \frac{1}{2}E_t\sum_{j=1}^{\infty}\beta^j \\ &= \frac{\beta}{2(1-\beta)} \end{aligned}$$

Let us assume the linear form of the second tree's pricing function.

$$p_{2t} = k_{2t}d_t$$

Next apply the above linear form to the Euler equation.

$$\begin{aligned} k_{2t}d_t &= E_t\beta\frac{d_t}{d_{t+1}}(d_{2t+1} + k_{2t+1}d_{t+1}) + \beta\frac{\gamma d_t}{s_{2t+1}} \\ k_{2t} &= E_t\beta k_{2t+1} + E_t\beta\left(\frac{d_{2t+1}}{d_{t+1}} + \frac{\gamma}{s_{2t+1}}\right) \end{aligned}$$

Recursively we obtain k_{2t}.

$$k_{2t} = E_t\sum_{j=1}^{\infty}\beta^j\left(\frac{d_{2t+j}}{d_{t+j}} + \frac{\gamma}{s_{2t+1}}\right)$$

Since we are given $d_{2t} = \frac{1}{2}d_t$ and $s_{2t+1} = 1$ in equilibrium.

$$\begin{aligned} k_{2t} &= E_t\sum_{j=1}^{\infty}\beta^j\left(\frac{1}{2} + \gamma\right) \\ &= \left(\frac{1}{2} + \gamma\right)\frac{\beta}{1-\beta} \end{aligned}$$

Finally, we have got the pricing equations.

$$\begin{aligned} p_{1t} &= \left(\frac{\beta}{2(1-\beta)}\right)d_t \\ p_{2t} &= \left(\left(\frac{1}{2} + \gamma\right)\frac{\beta}{1-\beta}\right)d_t \end{aligned}$$

(d) Prove that if $\gamma > 0$, then $R_{1t} > R_{2t}\ \forall\ t$

The returns R_{1t}, R_{2t} are defined as:

$$R_{1t} = \frac{p_{1t+1} + d_{1t+1}}{p_{1t}}$$

$$R_{2t} = \frac{p_{2t+1} + d_{2t+1}}{p_{2t}}$$

From the derived pricing equations,

$$\begin{aligned} R_{1t} &= \frac{\frac{1}{2}\left(\frac{\beta}{1-\beta}\right) d_{t+1} + \frac{1}{2} d_{t+1}}{\frac{1}{2}\left(\frac{\beta}{1-\beta}\right) d_t} \\ R_{2t} &= \frac{\left(\frac{1}{2}+\gamma\right)\left(\frac{\beta}{1-\beta}\right) d_{t+1} + \frac{1}{2} d_{t+1}}{\left(\frac{1}{2}+\gamma\right)\left(\frac{\beta}{1-\beta}\right) d_t} \end{aligned}$$

By rearranging the equations,

$$R_{1t} - R_{2t} = \left(1 - \frac{1}{1+2\gamma}\right) \frac{1-\beta}{\beta} \frac{d_{t+1}}{d_t}$$

If $\gamma > 0$, then

$$R_{1t} > R_{2t}$$