Borrowing Constraints

Financial Economics II

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1. **(Equilibrium with Exogenous Borrowing Constraints)** Consider an economy with a continuum [0,1] of consumers of symmetric types who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

Consumers of type 1 have an endowment steam of the single good in each period $(w_0^1, w_1^1, w_2^1, \ldots) = (8, 1, 8, 1, \ldots)$ while consumers of type 2 have $(\omega_0^2, \omega_1^2, \omega_2^2, \ldots) = (1, 8, 1, 8, \ldots)$. In addition there is one unit of trees that produce $d = 1$ units of good at each period. Each consumer of type i owns $\bar{s}_0^i$ of such a trees in period $t = 0$, $\bar{s}_0^i > 0$, $\bar{s}_0^1 + \bar{s}_0^2 = 1$. Trees do not grow or decay over time.

(a) Define an Arrow-Debreu equilibrium

Definition The following two equations are called the *good and financial market clear conditions*

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \ \forall t \tag{1}$$

$$s_t^1 + s_t^2 \leq 1 \ \forall t \tag{2}$$

Definition *A household problem* is defined by each agent's utility optimization problem:

$$\max \sum_{t=0}^{\infty} \beta^t \, u(c_t^i) \tag{3}$$

Subject to the following budget constraint

$$c_t^i + q_t s_{t+1}^i \leq \omega_t^i + (q_t + d)\, s_t^i \ \forall t \tag{4}$$

Definition An *Arrow-Debreu equilibrium* is an allocation $\{\{c^i, s_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^2$ and a sequence of prices $\{q_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem and satisfies the market clear conditions.

Decide optimal consumptions

From symmetric allocation, we can find optimal consumptions:

$$c^* = \widehat{c}_t^{\,1} = \widehat{c}_t^{\,2} = \frac{\omega^g + \omega^b + d}{2} = \frac{8 + 1 + 1}{2} = 5 \ \forall t \tag{5}$$

$$s_t^1 + s_t^2 \leq 1 \ \forall t \tag{6}$$

Compute the equilibrium prices

Using the Lagrange Multiplier,

$$L = \max \sum_{t=0}^{\infty} \beta^t \left(u(c_t^i) + \lambda_t(\omega_t^i + (q_t + d)\, s_t^i - c_t^i - q_t s_{t+1}^i) \right)$$

The following first-order conditions are obtained.

$$u'(c_t) + \lambda_t = 0$$

$$\beta \lambda_{t+1}(q_{t+1} + d) - \lambda_t q_t = 0$$

Using the above equations, we have reached the Euler Equation.

$$\frac{u'(c_t)}{\beta\, u'(c_{t+1})} = \frac{q_{t+1} + d}{q_t}$$

Since $u'(c^g) = u'(c^b)$ and prices are independent of time,

$$\frac{u'(c^g)}{\beta\, u'(c^b)} = \frac{q + d}{q}$$

$$\frac{1}{\beta} = \frac{q + d}{q}$$

Finally the equilibrium prices are obtained using $d = 1$.

$$q = \frac{\beta}{1-\beta} d = \frac{\beta}{1-\beta} \tag{7}$$

<u>Find the initial asset holdings</u>

From the budget constraint,

$$(q + d)s^b - qs^g = \frac{\omega}{2} - \omega^g$$

$$(q + d)s^g - qs^b = \frac{\omega}{2} - \omega^b$$

At the initial time, the agent 1 is in good shock and the agent 2 is in bad shock,

$$\bar{s}_0^1 = s^g$$

$$\bar{s}_0^2 = s^b$$

Next, the budget constraints can be written in a matrix form.

$$\begin{bmatrix} q + d & -q \\ -q & q + d \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} \frac{\omega}{2} - \omega^g \\ \frac{\omega}{2} - \omega^b \end{bmatrix} \tag{8}$$

This is a system of linear equations. Given conditions, $d = 1$, $\omega^g = 8$ and $\omega^b = 1$, we can solve this linear system.

$$\begin{bmatrix} q + d & -q \\ -q & q + d \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} 5 - 8 \\ 5 - 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{1-\beta} & -\frac{\beta}{1-\beta} \\ -\frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

We solved for $\bar{s}_0^1$ and $\bar{s}_0^2$

$$\begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\beta} & -\frac{\beta}{1-\beta} \\ -\frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \tag{9}$$

$$
\begin{bmatrix} \bar{s}_0^2 \\ \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+\beta} & \dfrac{\beta}{1+\beta} \\ \\ \dfrac{\beta}{1+\beta} & \dfrac{1}{1-\beta} \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix}
$$

$$
\begin{bmatrix} \bar{s}_0^2 \\ \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} \dfrac{4\beta-3}{1+\beta} \\ \\ \dfrac{4-3\beta}{1+\beta} \end{bmatrix} \tag{10}
$$

I have reached the initial asset holdings:

$$
\bar{s}\,_0^2 = \frac{4\beta-3}{1+\beta}
$$

$$
\bar{s}_0^1 = \frac{4-3\beta}{1+\beta}
$$

(b) Suppose that consumers cannot borrow or lend. Define an equilibrium for this liquidity constraint economy.

The crucial feature of the liquidity model is that physical capital cannot be held in negative amounts, and that there are no other securities that can be traded besides physical capital. Therefore, I want to add nonnegativity of physical capital holdings to the Arrow-Debreu market.

Definition The following two equations are called the *good and financial market clear conditions*

$$
c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \ \forall t \tag{11}
$$

$$
s_t^1 + s_t^2 \leq 1 \ \forall t \tag{12}
$$

Definition *A household problem* is defined by each agent's utility optimization problem:

$$
\max \sum_{t=0}^{\infty} \beta^t \, u(c_t^i) \tag{13}
$$

Subject to the following budget constraint

$$
c_t^i + q_t s_{t+1}^i \leq \omega_t^i + (q_t + d) \, s_t^i \ \forall t \tag{14}
$$

$$
s_t^i \geq 0
$$

Definition An *Arrow-Debreu equilibrium* is an allocation $\{\{c^i, s_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{q_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem and satisfies the market clear conditions.

(c) Compute the stochastic discount factor for this economy and compare it with the complete markets solution

Before starting this question, I want to make it clear that the stochastic discount factor is the price of pure discount bond so that it has a negative relationship with corresponding interest rate. We can switch from one to the other using the following equation:

$$
m = \frac{1}{1+r}
$$

where m and r mean stochastic discount factor and interest rate respectively.

The case of complete market

The stochastic discount factor is determined by the Euler equation.

$$m^c = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

where the superscript c indicates the complete market.

In the case of complete market, we can get a perfect risk sharing, meaning that consumptions are constant across periods.

$$c_t = c_{t+1} = c_{t+2} = ...$$

Hence, the stochastic discount factor in the complete market is derived as:

$$m^c = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \beta$$

The case of liquidity constrained market

In the liquidity constrained market, the stochastic discount factor is determined by the agents with good shock.

$$m^l = \frac{\beta u'(c^b)}{u'(c^g)}$$

where the superscript l indicates the complete market.

In the liquidity constrained market, the consumption with good shock is greater than or equal to the consumption with bad shock.

$$c^g \geq c^b$$

Since the marginal utility is decreasing,

$$u'(c^g) \leq u'(c^b)$$

We have arrived at the stochastic discount factor in the liquidity constrained market.

$$m^l = \frac{\beta u'(c^b)}{u'(c^g)} \geq \beta$$

$$m^l \geq m^c$$

The conclusion is that the stochastic discount factor is bigger in the liquidity constrained market than in the complete market. More exactly, if the liquidity constraint does not bind, two stochastic discount factors will be the same. Otherwise, they could be different. I would like to mention that the interest rate is smaller in the liquidity constrained market than in the complete market because interest rate is negatively related to the stochastic discount factor.

(d) Consider the function

$$F^L(c^g) = u'(c^g)(c^g - 8) + \beta\, u'(10 - c^g)(9 - c^g) \tag{15}$$

The case of $\beta = 0.9$

Show that when $\beta = 0.9$, $F^L(c^g) > 0$ and that $\widehat{c}_t^i = 5$ satisfies all of the equilibrium conditions for the liquidity constraint economy for the right choice of $\bar{s}_0^i$.

$$\begin{aligned} F^L(c^g) &= u'(c^g)(c^g - 8) + \beta\, u'(10 - c^g)(9 - c^g) \\ &= \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \\ &= \frac{5 - 8}{5} + \frac{(.9)(9 - 5)}{10 - 5} \\ &= 0.12 \end{aligned} \tag{16}$$

I have proved that $F^L(c^g) = .12 > 0$ when $\widehat{c}_t^i = 5$ and $\beta = .9$. Next let me show that those consumptions are in equilibrium. To prove this, we need to find price system and securities holdings and check whether or not they satisfy the budget constraint, the Euler equation, and the market clearing conditions.

From the budget constraints,

$$s^b = \frac{4\beta - 3}{1 + \beta} = \frac{4 \times 0.9 - 3}{1 + 0.9} = \frac{0.6}{1.9} = 0.316$$

$$s^g = \frac{4 - 3\beta}{1 + \beta} = \frac{-3 \times 0.9 + 4}{1 + 0.9} = \frac{1.3}{1.9} = 0.684$$

From the pricing equations,

$$q = \frac{\beta}{1 - \beta} = \frac{0.9}{1 - 0.9} = 9 \tag{17}$$

Equilibrium solution

$$\begin{aligned} q &= 9 \\ s^b &= 0.316 \\ s^g &= 0.684 \\ c^b &= c^g = 0.5 \end{aligned}$$

This equilibrium solution is calculated such that it satisfies all of the equilibrium conditions: the budget constraint, the Euler equation, and the market clearing conditions.

The case of $\beta = 0.2$

Show that when $\beta = 0.2$, the solution of $F^L(c^g) = 0$ and $c^g \in [5, 8]$ is such that

$$\widehat{c}_t^i = \begin{cases} c^g & \text{if } \omega_t^i = 8 \\ 10 - c^g & \text{if } \omega_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{s}_0^i$.

Since $u(c^i) = \log(c_t^i)$ and $\beta = 0.2$,

$$F^L(c^g) = \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \tag{18}$$

Setting $F^L(c^g) = 0$

$$F^L(c^g) = \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} = 0$$

$$\frac{c^g - 8}{c^g} = \frac{(c^g - 9)}{5(10 - c^g)}$$
$$5(c^g - 8)(10 - c^g) - c^g(c^g - 9) = 0$$
$$6(c^g)^2 - 99c^g + 400 = 0$$

By the quadratic formula, we solve for c^g.

$$c^g = \frac{99 \pm \sqrt{(99)^2 - 4(6)(400)}}{12}$$

The following solutions have been obtained.

$$c^g = 7.069 \quad \text{or} \quad 9.43 \tag{19}$$

The choice of the first solution $c^g = 7.069$ gives us $F^L(c^g) = 0$ and $c^g \in [5, 8]$

Like the previous case, we need to find price system and securities holdings and check whether or not they satisfy the budget constraint, the Euler equation, and the market clearing conditions. Since the liquidity constraint binds in this case, the securities holdings are determined as:

$$s^b = 0$$

$$s^g = 1$$

From the Euler equation of agent with good shock,

$$\begin{aligned}
\frac{\beta u'(c^b)}{u'(c^g)} &= \frac{q}{q+d} \\
\frac{\beta c^g)}{u'(c^b)} &= \frac{q}{q+d} \\
qc^b &= 0.2c^g(q+d) \\
q &= \frac{(0.2)c^g}{c^b - 0.2c^g} = 0.9318
\end{aligned}$$

Equilibrium solution

$$\begin{aligned}
q &= 0.9318 \\
s^b &= 0 \\
s^g &= 1 \\
c^b &= 2.931 \\
c^g &= 7.069
\end{aligned}$$

This equilibrium solution is calculated such that it satisfies all of the equilibrium conditions: the budget constraint, the liquidity constraint, and the market clearing conditions.

2. **(Equilibrium with Endogenous Borrowing Constraints)** Consider the economy from the previous question. However, we assume that consumers face borrowing constraints of the form

$$\sum_{j=0}^{\infty} \beta^i \log c_{t+j}^i \geq \sum_{j=0}^{\infty} \beta^i \log \omega_{t+j}^i \ \forall t, i$$

but otherwise the definition of equilibrium is the same as in part (a).

(a) Provide a motivation for this environment and explain the economic initution of the constraint.

At any point in time, households have an incentive to renege on their claims and walk away from the credit market. If they choose to default, they are excluded from all further participation in intertemporal trade. However, they still have claim on endowment from human capital as we consider the human capital as inalienable. This model requires some central authority to keep track of who has gone bankrupt and to assure that their physical capital is seized. If they decide to default, households consume like autarky economy because they cannot lend nor borrow any more from financial markets. Therefore, the households face the following rationality constraint:

$$(1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau^i) \geq (1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(\omega_\tau^i)$$

This says that in every period, the value of continuing to participate in the economy is no less than the value of dropping out. The credit agency will never lend so much to consumers that they will choose bankruptcy.

(b) Define an equilibrium for this debt constraint economy.

Definition The following equation is called the *good and financial market clear condition*

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \ \forall t \tag{20}$$

$$s_t^1 + s_t^2 \leq 1 \ \forall t \tag{21}$$

Definition *A household problem* is defined by each agent's utility optimization problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{22}$$

Subject to the following budget constraint and rationality constraint

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t(\omega_t^i + s_0^i d) \tag{23}$$

$$(1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau^i) \geq (1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(\omega_\tau^i) \ \forall \ t$$

Definition An *equilibrium in the debt constraint economy* is an allocation $\{\{c_t^i, s_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{p_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem and satisfies the market clear condition.

(c) Consider the function

$$F^D(c^g) = u(c^g) - u(8) + \beta[u(10 - c^g) - u(1)] \tag{24}$$

The case of $\beta = 0.9$

Show that when $\beta = 0.9$, $F^D(c^g) > 0$ and that $\widehat{c}_t^i = 5$ satisfies all of the equilibrium conditions for the debt constraint economy.

$$\begin{aligned}
F^D(c^g) &= u(c^g) - u(8) + \beta\,(u(10 - c^g) - u(1)) \\
&= u(5) - u(8) + (0.9)(u(10 - 5) - u(1)) \\
&= \log(5) - \log(8) + (0.9)(\log(5) - \log(1)) \\
&= 0.9785
\end{aligned} \tag{25}$$

I have proved that $F^D(c^g) = 0.9785 > 0$ when $\widehat{c}_t^i = 5$ and $\beta = .9$. Next let me show that those consumptions are in equilibrium. To prove this, we need to find price system and check whether or not they satisfy the budget constraint, the Euler equation, and the market clearing conditions.

From the Euler equation of agent with good shock,

$$\begin{aligned}
\frac{\beta u'(c^b)}{u'(c^g)} &= \frac{q}{q+d} \\
\beta &= \frac{q}{q+d} \\
q &= (0.9)(q+d) \\
(0.1)q &= (0.9) \\
q &= 9
\end{aligned}$$

Equilibrium solution

$$\begin{aligned}
q &= 9 \\
c^b &= 5 \\
c^g &= 5
\end{aligned}$$

This equilibrium allocation is calculated such that it satisfies all of the equilibrium conditions: the budget constraint, the Euler equation, and the market clearing conditions.

The case of $\beta = 0.2$

Show that when $\beta = 0.2$, the solution of $F^D(c^g) = 0$ and $c^g \in [5, 8]$ is such that

$$\widehat{c}_t^i = \begin{cases} c^g & \text{if } \omega_t^i = 8 \\ 10 - c^g & \text{if } \omega_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{s}_0^i$. If $\beta = .2$, the solution to $F^D(c^g) = 0$ and $c^g \in [5, 8]$ is approximately 6.09076.

Since $u(c^i) = \log(c_t^i)$ and $\beta = 0.2$,

$$F^D(c^g) = \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \tag{26}$$

Setting $F^D(c^g) = 0$

$$
\begin{aligned}
F^D(c^g) &= \frac{c^g - 8}{c^g} + \frac{\beta(9 - c^g)}{10 - c^g} = 0 \\
\log(c^g) - \log(8) + (0.2)(\log(10 - c^g) - \log(1)) = 0 \\
\log(c^g) - \log(8) + (0.2)\log(10 - c^g) = 0 \\
\log(c^g)^5(10 - c^g) = \log(8^5) \\
(c^g)^5(10 - c^g) = (8^5) \\
(c^g)^6 - 10(c^g)^5 + 8^5 = 0
\end{aligned}
$$

Let us see if $c^g = 6.09076$ is a solution to the above equation. Since $(6.09076)^6 - 10(6.09076)^5 + 8^5 = 0$, $c^g = 6.09076$ is the solution to $F^D(c^g) = 0$.

Like the previous case, we need to find price system and check whether or not they satisfy the budget constraint, the Euler equation, and the market clearing condition. From the Euler equation of agent with good shock,

$$
\begin{aligned}
\frac{\beta u'(c^b)}{u'(c^g)} &= \frac{q}{q+d} \\
\frac{\beta c^g)}{u'(c^b)} &= \frac{q}{q+d} \\
qc^b &= 0.2c^g(q+d) \\
q &= \frac{(0.2)c^g}{c^b - 0.2c^g} = 0.452
\end{aligned}
$$

Equilibrium solution

$$
\begin{aligned}
q &= 0.452 \\
c^b &= 3.90924 \\
c^g &= 6.09076
\end{aligned}
$$

This equilibrium allocation is calculated such that it satisfies all of the equilibrium conditions: the budget constraint, the debt rationality constraint, and the market clearing conditions.

(d) Compute the stochastic discount factor for this economy and compare it with the liquidity constraint model. Do you think these two economies can predict the same pattern for asset price movements?

Let me start with the final conclusion. The stochastic discount factor in the debt constraint economy is smaller than that in the liquidity constraint economy. Regarding interest rates, the interest rate in the debt constraint economy is larger than that in the liquidity constraint economy because the interest rates are negatively related to the stochastic discount factor.

The reason is that the liquidity constraint has less consumption smoothing and the debt constraint exhibits a large degree of consumption smoothing as seen in the previous questions 1(d) and 2(c). The stochastic discount factor is derived by the following equation:

$$
m = \frac{\beta u'(c^b)}{u'(c^g)}
$$

As more consumption smoothing is obtained in the debt constraint economy, c^b and c^g are closer to each other and so the stochastic discount factor gets closer to β. However, in the liquidity constraint economy,c^b and c^g keep away from each other and so the stochastic discount factor can be much bigger than β. Therefore,the stochastic discount factor in the debt constraint economy is smaller than that in the liquidity constraint economy. The difference of stochastic discount factors leads to different asset prices.

(e) Unfortunately, the value of $\bar{s}_0^i$ that you calculated in the previous section when $\beta = 0.2$ is negative. Can you think of another way to make the proposed steady state an equilibrium? Explain.

In these borrowing constraint economy, the consumer who first has good productivity has a permanent advantage over the other. The easiest way to compensate for this advantage and arrive at the steady state is to impose a transfer payment from one consumer type to the other. In the liquidity model, we need the budget constraint for the consumer who first has high productivity in the first period.

$$c^g + q \leq \omega^g + s_0^1(q+d) - \tau$$

In the debt constraint economy, we need to transfer enough income so that the present discounted value of lifetime incomes are equal.

$$\sum_{t=0}^{\infty} p_t(\omega_t^1 + \theta_0^1 d) - \tau = \sum_{t=0}^{\infty} p_t(\omega_t^2 + \theta_0^2 d) + \tau$$

Alternatively, in the debt constraint economy, we could introduce uncertainty before the first period, giving both consumer types equal chances of having the high productivity first, and allowing them to write contingent contracts against this initial uncertainty.

(f) Write down the endogenous borrowing constraint if the agent is allowed to save in a storage technology with a return $R > 0$, following a default period.

If there is storage technology, we cannot use autarky economy any more. In this case, we need to know optimal storage policy if we decide to default. Let me denote v^{str} the optimal utility in the storage economy when we decide to default.

$$\begin{aligned} v^{str} &= \max\ u(w_t - s_t) + \beta u(w_{t+1} + s_t R - s_{t+1}) + \beta^2 u(w_{t+2} + s_{t+1}R - s_{t+2}) + ... \\ &= \max\ u(w_t - s_t) + \sum_{j=1}^{\infty} \beta^j u(w_{t+j} + s_{t+j-1}R - s_{t+j}) \end{aligned}$$

where s_{t+j} means the amount of savings at time t+j.

This optimal problem can be solved using Cake Eating Problem. The first order condition can be obtained taking differentiation with respect to s_{t+j}.

$$-\beta^j u'(w_{t+j} + s_{t+j-1}R - s_{t+j}) + \beta^{j+1} R u'(w_{t+j+1} + s_{t+j}R - s_{t+j+1})$$
$$u'(w_{t+j} + s_{t+j-1}R - s_{t+j}) = \beta R u'(w_{t+j+1} + s_{t+j}R - s_{t+j+1})$$

From the Euler equation, the amount of savings can be determined and finally the optimal storage utility v^{str} can be found. Once we find v^{str}, the rationality constraint will be as follows:

$$\sum_{j=0}^{\infty} \beta^j u(c_{t+j}^i) \geq v^{str}$$

where

$$v^{str} = \max u(w_t - s_t) + \sum_{j=1}^{\infty} \beta^j u(w_{t+j} + s_{t+j-1}R - s_{t+j})$$

(g) Bankruptcy law only precludes individuals from trading just for a finite number of periods. How would you modify this constraint to accommodate this institutional feature.

Let the finite number of periods be T. During the T period, households consume like autarky economy and after that they can access original debt constraint economy. In this case I want to calculate the maximum utility that they will receive in case of default.

$$v^{max} = \sum_{j=0}^{T} \beta^j u(w^i_{t+j}) + v^{aut}$$

The utility after T period can be calculated as follows:

$$v^{aut} = \sum_{j=0}^{\infty} \beta^{T+j} u(c^i_{t+T+j})$$

Subject to the following budget constraint and rationality constraint

$$\sum_{j=0}^{\infty} p_{T+j}\, c^i_{T+j} = \sum_{j=0}^{\infty} p_{T+j}(\omega^i_{T+j} + s^i_T d) \tag{27}$$

$$\sum_{j=0}^{\infty} \beta^{T+j} u(c^i_{T+j}) \geq \sum_{j=0}^{\infty} \beta^{T+j} u(\omega^i_{T+j}) \ \forall\ j$$

Once we find v^{aut}, the endogenous borrowing constraint will be like:

$$\sum_{j=0}^{\infty} \beta^j u(c^i_{t+j}) \geq \sum_{j=0}^{T} \beta^j u(w^i_{t+j}) + v^{aut}$$

3. **(Equilibrium with Endogenous Borrowing Constraints and Uncertainty)** We want to take the model one step further and include uncertainty along two dimensions. First, we assume that income shocks are persistent, and evolve according to a symmetric Markov process with a probability distribution

$$\prod_{\omega'|\omega} = \begin{bmatrix} \pi_{gg} & \pi_{bg} \\ \pi_{gb} & \pi_{bb} \end{bmatrix} = \begin{bmatrix} 1-\pi & \pi \\ \pi & 1-\pi \end{bmatrix}$$

Second we assume i.i.d. dividend shock where γ is the probability of receiving a good dividend shock $d^g = 3$, and $1-\gamma$ is the probability of receiving a bad dividend or aggregate shock, $d^b = 1$. Since we now have a aggregate uncertainty we can still solve for a symmetric equilibrium, but consumption across states of nature will differ due to aggregate uncertainty.

(a) Write down the market equilibrium in the endogenous borrowing constraint economy. [**Hint:** You should be very careful when writing the endogenous borrowing constraint, since it could depend on aggregate uncertainty.]

Definition The following equation is called the *good and financial market clear condition* in the case of aggregate uncertainty.

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d_t = \omega \ \forall t \tag{28}$$

$$s_t^1 + s_t^2 \leq 1 \ \forall t \tag{29}$$

In the good dividend shock,

$$c^{gg} + c^{bg} = \omega^g + \omega^b + d^g = \bar{\omega}$$

In the bad dividend shock,

$$c^{gb} + c^{bb} = \omega^g + \omega^b + d^b = \underline{\omega}$$

where the first superscript means the status of endowment shock while the second superscript indicates the status of dividend shock.

Definition *A household problem* is defined by each agent's utility optimization problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{30}$$

Subject to the following budget constraint and rationality constraint

$$E_0 \sum_{t=0}^{\infty} p_t c_t^i = E_0 \sum_{t=0}^{\infty} p_t(\omega_t^i + s_0^i d_t) \tag{31}$$

$$(1-\beta)E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}^i) \geq (1-\beta)E_t \sum_{j=0}^{\infty} \beta^j u(\omega_{t+j}^i) \ \forall \ t$$

where I would like to mention that the dividend in this budget constraint is time-varying.

Definition An *equilibrium in the debt constraint economy* is an allocation $\{\{c_t^i, s_t^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{p_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem and satisfies the market clear condition.

(b) Solve for the equilibrium allocations when the borrowing constraint binds, and calculate the stochastic discount factor in the presence of aggregate shocks. Will the borrowing constraint always bind? [**Hint:** You can use the social planner problem to solve for efficient allocations.]

In this case, since there is aggregate uncertainty, the borrowing constraint depends on the status of dividend shock.

The case of good dividend shock

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}^i) &= u(c^{gg}) + \sum_{j=1}^{\infty} \beta^j u((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb})) \\ &= u(c^{gg}) + \frac{\beta((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb}))}{1-\beta} \end{aligned}$$

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j u(\omega_{t+j}^i) &= u(\omega^g) + \sum_{j=1}^{\infty} \beta^j ((1-\pi)u(\omega^g) + \pi u(\omega^b)) \\ &= u(\omega^g) + \frac{\beta((1-\pi)u(\omega^g) + \pi u(\omega^b))}{1-\beta} \end{aligned}$$

Therefore, we can obtain the rationality constraint as follows:

$$u(c^{gg}) + \frac{\beta((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb}))}{1-\beta}$$

$$\geq u(\omega^g) + \frac{\beta((1-\pi)u(\omega^g) + \pi u(\omega^b))}{1-\beta}$$

$$F(c^{gg}, c^{gb}) = (1-\beta)(u(c^{gg}) - u(\omega^g))$$
$$+\beta(((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb})) - ((1-\pi)u(\omega^g) + \pi u(\omega^b)))$$

$$F(c^{gg}, c^{gb}) = (1-\beta)(u(c^{gg}) - u(\omega^g)) + \beta(1-\pi)(\gamma u(c^{gg}) + (1-\gamma)u(c^{gb}) - u(\omega^g))$$
$$+\beta(\pi)(\gamma u(c^{bg}) + (1-\gamma)u(c^{bb}) - u(\omega^b))$$

Since $c^{bg} = \bar{\omega} - c^{gg}$ and $c^{bb} = \underline{\omega} - c^{gb}$

$$F(c^{gg}, c^{gb}) = (1-\beta)(u(c^{gg}) - u(\omega^g)) + \beta(1-\pi)(\gamma u(c^{gg}) + (1-\gamma)u(c^{gb}) - u(\omega^g))$$
$$+\beta(\pi)(\gamma u(\bar{\omega} - c^{gg}) + (1-\gamma)u(\underline{\omega} - c^{gb}) - u(\omega^b))$$

The case of bad dividend shock

Similarly we can calculate the rationality consraint with bad dividend shock.

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}^i) &= u(c^{gb}) + \sum_{j=1}^{\infty} \beta^j u((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb})) \\ &= u(c^{gb}) + \frac{\beta((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb}))}{1-\beta} \end{aligned}$$

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j u(\omega_{t+j}^i) &= u(\omega^g) + \sum_{j=1}^{\infty} \beta^j ((1-\pi)u(\omega^g) + \pi u(\omega^b)) \\ &= u(\omega^g) + \frac{\beta((1-\pi)u(\omega^g) + \pi u(\omega^b))}{1-\beta} \end{aligned}$$

Noting that the first term in the right hand side only is different from the previous equation, we can get to the following equation:

$$G(c^{gg}, c^{gb}) = (1-\beta)(u(c^{gb}) - u(\omega^g)) + \beta(1-\pi)(\gamma u(c^{gg}) + (1-\gamma)u(c^{gb}) - u(\omega^g))$$
$$+\beta(\pi)(\gamma u(\bar{\omega} - c^{gg}) + (1-\gamma)u(\underline{\omega} - c^{gb}) - u(\omega^b))$$

Is the borrowing constraint always binding?

In this question, the parameters are not given. I am going to see what will happen in the extreme case. For this purpose, I would like to see how the parameters affect default decision.

As β is decreasing, the value of current consumption is getting more valuable than that of future consumption. It means that as β is decreasing, the household is likely to choose to default. How about γ? As γ is decreasing, the household will get less dividend so that he is more likely to choose to default. In the case of π, as π is increasing, the household has less persistence in endowment shock. It means that if he has good shock today, then he will get bad shock tomorrow. Hence, more likely, he is going to choose to default today. To sum up, the following parameters are such that the household is less likely to default.

$$\beta = 1$$

$$\gamma = 1$$

$$\pi = 0$$

If the household will default in this extreme case, he will want to default for any set of parameters. Under the given parameters, $F(c^{gg}, c^{gb})$ and $G(c^{gg}, c^{gb})$ are reduced to:

$$F(c^{gg}, c^{gb}) = u(c^{gg}) - u(\omega^g)$$

$$G(c^{gg}, c^{gb}) = u(c^{gg}) - u(\omega^g)$$

As a matter of fact, the second function has no meaning because when $\gamma = 1$ the household receives only good dividend. From the first solution,

$$c^{gg} = \omega^g = 9$$

From the good market clearing condition,

$$c^{bg} = \bar{\omega} - c^{gg} = 3$$

The solution $c^{gg} = 9$ and $c^{bg} = 3$ means that the household will choose to default in this extreme case. Hence I draw a final conclusion that the borrowing constraint will bind for any set of parameters.

Solve for equilibrium allocations

Since the borrowing constraint is always binding, $G(c^{gg}, c^{gb}) = 0$. It means that households get an imperfect risk sharing when the dividend is bad, while they get perfect risk sharing when the dividend is good. Hence, we can decide the optimal consumptions as follows:

$$c^{gg} = c^{bg} = \frac{\bar{\omega}}{2}$$

Now that we know c^{gg}, we can calculate c^{gb} from the condition $G(c^{gg}, c^{gb}) = 0$. Finally, it gives the value c^{bb} from the equation $c^{gb} + c^{bb} = \underline{\omega}$.

Calculate the stochastic discount factor

The stochastic discount factor is determined by the marginal rates of substitutions of the household with good endowment shock. But since dividend is stochastic, the stochastic discount factor will also be stochastic. That is, the stochastic discount factor depends on the current status of dividend shock.

In the case of good dividend

$$\begin{aligned} m^g &= \frac{E_t u(c_{t+1})}{u(c_t)} \\ &= \frac{\beta((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb}))}{u^{gg}} \end{aligned}$$

In the case of bad dividend

$$\begin{aligned} m^b &= \frac{E_t u(c_{t+1})}{u(c_t)} \\ &= \frac{\beta((1-\pi)\gamma u(c^{gg}) + (\pi)\gamma u(c^{bg}) + (1-\pi)(1-\gamma)u(c^{gb}) + (\pi)(1-\gamma)u(c^{bb}))}{u^{gb}} \end{aligned}$$